Lecture 30:
The Cauchy Integral Formula

Dan Sloughter
Furman University
Mathematics 39

April 28, 2004

30.1 The Cauchy integral formula

Theorem 30.1. Suppose f is analytic in the region consisting of a simple closed contour C, positively oriented, and all points in the interior of C. If z_0 is in the interior of C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

Proof. Choose $\rho > 0$ such that the circle C_ρ of radius ρ with center z_0 lies entirely within the interior of C. Take C_ρ with positive orientation. Since

$$\frac{f(z)}{z - z_0}$$

is analytic on C, C_ρ, and in the region between C and C_ρ, we have

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_\rho} \frac{f(z)}{z - z_0} dz.$$

Hence

$$\begin{aligned} \int_C \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_\rho} \frac{1}{z - z_0} dz &= \int_{C_\rho} \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_\rho} \frac{1}{z - z_0} dz \\ &= \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz. \end{aligned}$$

Now

$$\int_{C_\rho} \frac{1}{z - z_0} dz = 2\pi i,$$

so we have

$$\int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

It remains to show that the integral on the right is 0.

Let $\epsilon > 0$. Since f is continuous at z_0, there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon$$

whenever $|z - z_0| < \delta$. If we choose $\rho < \delta$, then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \frac{|f(z) - f(z_0)|}{|z - z_0|} < \frac{\epsilon}{\rho}$$

for all $z \in C_\rho$. Hence

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \right| < \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

□

Example 30.1. If C is the circle $|z| = 1$ with positive orientation, then

$$\int_C \frac{e^z}{z} = 2\pi i e^0 = 2\pi i.$$

Example 30.2. If C is the circle $|z| = 2$ with positive orientation, then

$$\begin{aligned} \int_C \frac{1}{z^2 + 1} dz &= \int_C \frac{1}{(z + i)(z - i)} dz \\ &= \int_{C_1} \frac{\frac{1}{z+i}}{z - i} dz + \int_{C_2} \frac{\frac{1}{z-i}}{z + i} dz \end{aligned}$$

$$
\begin{aligned}
&= 2\pi i \frac{1}{i+i} + 2\pi i \frac{1}{-i-i} \\
&= \pi - \pi \\
&= 0,
\end{aligned}
$$

where C_1 is the circle $|z - i| = \frac{1}{2}$ and C_2 is the circle $|z + i| = \frac{1}{2}$.