Lecture 26:
The Modulus of an Integral

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26.1 An upper bound

Proposition 26.1. If $|f(z)| \leq M$ for all $z \in C$, where C is a contour $z(t)$, $a \leq t \leq b$, and L is the length of C, then

$$\left| \int_C f(z)dz \right| \leq ML$$

Proof. Since

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt,$$

we have

$$\begin{aligned} \left| \int_C f(z)dz \right| &\leq \int_a^b |f(z(t))||z'(t)|dt \\ &\leq \int_a^b M|z'(t)|dt \\ &= M \int_a^b |z'(t)|dt \\ &= ML. \end{aligned}$$

□

Note that such an M always exists since we assume that $f(z(t))$ is a piecewise continuous function on a closed interval $[a, b]$ (this is the extreme value theorem from calculus).

Example 26.1. Consider

$$\int_C \frac{z^2+1}{z^6+1} dz$$

where C is the arc of the circle $|z| = 3$ from 3 to -3. Now for z on C,

$$|z^2+1| \leq |z^2| + |1| = |z|^2 + 1 = 9 + 1 = 10$$

and

$$|z^6+1| \geq \big||z|^6 - |1|\big| = 728.$$

Hence

$$\left|\frac{z^2+1}{z^6+1}\right| = \frac{|z^2+1|}{|z^6+1|} \leq \frac{10}{728} = \frac{5}{364}.$$

Since C has length 3π, it follows that

$$\left|\int_C \frac{z^2+1}{z^6+1} dz\right| \leq \frac{15\pi}{364}.$$

Example 26.2. Now consider

$$\int_C \frac{z^2+1}{z^6+1} dz$$

where C is the arc of the circle $|z| = R$ from R to $-R$, $R > 1$. We have, for z on C,

$$|z^2+1| \leq R^2 + 1$$

and

$$|z^6+1| \geq R^6 - 1,$$

and so

$$\left|\frac{z^2+1}{z^6+1}\right| \leq \frac{R^2+1}{R^6-1}.$$

Thus

$$\left|\int_C \frac{z^2+1}{z^6+1} dz\right| \leq \frac{(R^2+1)R\pi}{R^6-1}.$$

Now

$$\lim_{R\to\infty} \frac{(R^2+1)R\pi}{R^6-1} = \lim_{R\to\infty} \frac{\frac{\pi}{R^3}+\frac{\pi}{R^5}}{1-\frac{1}{R^6}} = 0,$$

and so

$$\lim_{R\to\infty} \left| \int_C \frac{z^2+1}{z^6+1} dz \right| = 0.$$

Hence

$$\lim_{R\to\infty} \int_C \frac{z^2+1}{z^6+1} dz = 0.$$