Lecture 24:
Contours

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24.1 Curves

Definition 24.1. Suppose $x : [a, b] \to \mathbb{R}$ and $y : [a, b] \to \mathbb{R}$ are both continuous and let $z(t) = x(t) + iy(t)$. We call the set

$$C = \{w \in \mathbb{C} : w = z(t), a \leq t \leq b\}$$

an *arc*. We call C a *simple arc* if $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$, and we call C a *simple closed curve*, or a *Jordan curve*, if $z(b) = z(a)$ and $z(t_1) \neq z(t_2)$ whenever $a < t_1 < b$, $a < t_2 < b$, and $t_1 \neq t_2$.

To be precise, an arc is the set of points C along with the parametrization $z(t)$.

Example 24.1. The arc described by $z(t) = e^{it}$, $0 \leq t \leq 2\pi$, is the unit circle centered at the origin, and is a simple closed curve. The arc described by $w(t) = e^{-it}$, $0 \leq t \leq 2\pi$, is the same set of points, but is not the same as the previous arc because the parametrization is different.

Example 24.2. More generally, for any $z_0 \in \mathbb{C}$ and $R > 0$, the arc described by $z(t) = z_0 + Re^{it}$, $0 \leq t \leq 2\pi$, is a circle of radius R centered at z_0.

Example 24.3. Note that the arc described by $z(t) = e^{i2t}$, $0 \leq t \leq 2\pi$, is, as a set of points, the unit circle centered at the origin, but is not a simple closed curve since the circle is traversed twice as t goes from 0 to 2π.

24.2 Arclength

Suppose $z(t)$ describes an arc C for $a \le t \le b$. If we divide $[a, b]$ into n subintervals, each of length

$$\Delta t = \frac{b-a}{n}$$

with endpoints $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, then

$$\sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$$

approximates the length of the arc from $z(t_{i-1})$ to $z(t_i)$. If L is the length of C, then

$$\begin{aligned} L &\approx \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2} \\ &= \sqrt{\left(\frac{x(t_i) - x(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{y(t_i) - y(t_{i-1})}{\Delta t}\right)^2} \Delta t. \end{aligned}$$

Letting $n \to \infty$ (equivalently, $\Delta t \to 0$), we expect

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b |z'(t)| dt.$$

Example 24.4. If L is the length of the curve described by $z(t) = e^{it}$, $0 \le t \le 2\pi$, then $|z'(t)| = |ie^{it}| = 1$, and so

$$L = \int_0^{2\pi} 1 dt = 2\pi.$$

Now suppose $z(t)$, $a \le t \le b$, describes an arc C and $\varphi : [c, d] \to [a, b]$ maps $[c, d]$ onto $[a, b]$. Moreover, suppose φ is continuous on $[c, c]$, differentiable on (c, d), and $\varphi'(t) > 0$ for all $t \in (c, d)$. Then

$$Z(t) = z(\varphi(t)), c \le t \le d,$$

also describes (that is, parametrizes) the arc C. If L is the length of C, then, as described above,

$$L = \int_a^b |z'(t)| dt.$$

If we make the substitution $t = \varphi(s)$, then $dt = \varphi'(s)ds$, and so

$$L = \int_c^d |z'(\varphi(s))|\varphi'(s)ds = \int_c^d |Z'(s)|ds,$$

where we have used that fact that

$$|Z'(s)| = |z'(\varphi(s))\varphi'(s)| = |z'(\varphi(s))|\varphi'(s)$$

because of the chain rule and the fact that $\varphi'(s) > 0$ for all s. Hence, as we should expect, the length of an arc does not depend on the parametrization.

Example 24.5. Note that $Z(t) = e^{i2t}$, $0 \leq t \leq \pi$, describes the same set of points, namely, the unit circle centered at the origin, as in the previous example. This time $Z'(t) = 2ie^{i2t}$, and so $|Z'(t)| = 2$ and we find

$$L = \int_0^\pi 2dt = 2\pi.$$

Note, however, that if we had $0 \leq t \leq 2\pi$, then we would find

$$L = \int_0^{2\pi} 2dt = 4\pi$$

because this parametrization of the unit circle traverses the circle twice.

24.3 Smooth curves and contours

Suppose $z(t)$, $a \leq t \leq b$, describes an arc C and $z'(t) \neq 0$ for all $t \in (a, b)$. In multi-variable calculus, one interprets $z'(t)$ geometrically as a vector tangent to C at $z(t)$, and then defines

$$T = \frac{z'(t)}{|z'(t)|}$$

to be the unit tangent vector. If $z'(t)$ is continuous, then T varies continuously, and we think of the curve as being smooth.

Definition 24.2. We say an arc $z(t)$ is *smooth* if $z'(t)$ is continuous on $[a, b]$ and $z'(t) \neq 0$ for all $t \in (a, b)$.

We call a finite number of smooth arcs joined end to end a *contour*. If $z(t)$, $a \leq t \leq b$, parametrizes a contour C, then $z(t)$ is continuous and $z'(t)$ is piecewise continuous. Moreover, if $z(a) = z(b)$ but $z(t_1) \neq z(t_2)$ for all $t_1, t_2 \in (a, b)$, then we call C a *simple closed contour*.

The following result, the *Jordan curve theorem*, appears intuitively obvious, but is surprisingly hard to prove.

Theorem 24.1. If $z(t)$ parametrizes a simple closed contour C, then

$$\mathbb{C} = C \cup I \cup E$$

where (1) C, I, and E are disjoint; (2) I is bounded; (3) E is unbounded; and (4) C is the boundary of both I and E.

We call I the *interior* of C and E the *exterior* of C.