Lecture 23:
Complex Functions of a Real Variable

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23.1 Derivatives

Definition 23.1. Suppose $S \subset \mathbb{R}$ and $f : S \to \mathbb{C}$ is defined on an open interval containing the point t_0. If

$$f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$$

exists, we say f is *differentiable* at t_0 and we call $f'(t_0)$ the *derivative* of f at t_0.

Proposition 23.1. If $f : \mathbb{R} \to \mathbb{C}$ with $f(t) = u(t) + iv(t)$, then f is differentiable at t_0 if and only if u and v are differentiable at t_0, in which case

$$f'(t) = u'(t) + iv'(t).$$

Proof. The result follows from the observation that

$$\begin{aligned} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} &= \frac{(u(t_0 + \Delta t) + iv(t_0 + \Delta t)) - (u(t_0) + iv(t_0))}{\Delta t} \\ &= \frac{u(t_0 + \Delta t) - u(t_0)}{\Delta t} + i\frac{v(t_0 + \Delta t) - v(t_0)}{\Delta t}. \end{aligned}$$

□

Example 23.1. If
$$w(t) = \sin(3t) - i\cos(4t),$$
then
$$w'(t) = 3\cos(3t) + 4i\sin(4t).$$

The following results follow as before.

Proposition 23.2. Suppose $c \in \mathbb{C}$ is a constant, $U \subset \mathbb{R}$, and $w : U \to \mathbb{C}$ and $s : U \to \mathbb{C}$ are differentiable. Then
$$\frac{d}{dt}(cw(t)) = cw'(t),$$
$$\frac{d}{dt}(w(t) + s(t)) = w'(t) + s'(t),$$
$$\frac{d}{dt}w(t)s(t) = w(t)s'(t) + s(t)w'(t),$$
and, provided $s(t) \neq 0$,
$$\frac{d}{dt}\left(\frac{w(t)}{s(t)}\right) = \frac{s(t)w'(t) - w(t)s'(t)}{(s(t))^2}.$$

Proposition 23.3. If w is differentiable, then
$$\frac{d}{dt}e^{w(t)} = w'(t)e^{w(t)}.$$

Proof. If $w(t) = u(t) + iv(t)$, then
$$e^{w(t)} = e^{u(t)}\cos(v(t)) + ie^{u(t)}\sin(v(t)),$$

and so
$$\begin{aligned}
\frac{d}{dt}e^{w(t)} &= -e^{u(t)}\sin(v(t))v'(t) + e^{u(t)}\cos(v(t))u'(t) \\
&\qquad + i(e^{u(t)}\cos(v(t)v'(t) + e^{u(t)}\sin(v(t))u'(t)) \\
&= u'(t)e^{u(t)}(\cos(v(t) + i\sin(v(t))) + v'(t)e^{u(t)}(-\sin(v(t) + i\cos(v(t))) \\
&= u'(t)e^{u(t)}e^{iv(t)} + iv'(t)e^{u(t)}(\cos(v(t)) + i\sin(v(t))) \\
&= u'(t)e^{u(t)}e^{iv(t)} + iv'(t)e^{u(t)}e^{iv(t)} \\
&= (u'(t) + iv'(t))e^{u(t)+iv(t)} \\
&= w'(t)e^{w(t)}.
\end{aligned}$$

□

Example 23.2. We have

$$\frac{d}{dt}t^2e^{4it} = t^2(4ie^{4it}) + 2te^{4it} = (2t + 4it^2)e^{4it}.$$

23.2 Integration

Definition 23.2. Suppose $u : [a, b] \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$ are both integrable and let $w(t) = u(t) + iv(t)$. Then we call

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

the *definite integral* of w on $[a, b]$.

Example 23.3. If $w(t) = t^2 + i\sin(\pi t)$, then

$$\int_0^1 w(t)dt = \int_0^1 t^2dt + i\int_0^1 \sin(\pi t)dt = \frac{1}{3} + i\frac{2}{\pi}.$$

Note that if $W'(t) = w(t)$ and we let $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$, then $U'(t) = u(t)$ and $V'(t) = v(t)$, and so

$$\begin{aligned}\int_a^b w(t)dt &= \int_a^b u(t)dt + i\int_a^b v(t)dt \\ &= U(b) - U(a) + i(V(b) - V(a)) \\ &= (U(b) + iV(b)) - (U(a) + iV(a)) \\ &= W(b) - W(a).\end{aligned}$$

Example 23.4. It follows that

$$\int_0^2 e^{4it}dt = \frac{1}{4i}e^{4it}\Big|_0^2 = -\frac{i}{4}(e^{8i} - 1).$$

Proposition 23.4. If $w(t)$ is integrable on $[a, b]$, then

$$\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt.$$

Proof. The inequality clearly holds if

$$\int_a^b w(t)dt = 0.$$

So suppose

$$\int_a^b w(t)dt = r_0 e^{i\theta_0}$$

where $r_0 > 0$. Then

$$\left| \int_a^b w(t)dt \right| = r_0$$

and

$$\begin{aligned}
r_0 &= \frac{1}{e^{i\theta_0}} \int_a^b w(t)dt \\
&= \int_a^b e^{-i\theta_0} w(t)dt \\
&= \mathrm{Re} \int_a^b e^{-i\theta_0} w(t)dt \\
&= \int_a^b \mathrm{Re}\left(e^{-i\theta_0} w(t)\right) dt \\
&\leq \int_a^b \left|e^{-i\theta_0} w(t)\right| dt \\
&= \int_a^b \left|e^{-i\theta_0}\right| |w(t)| dt \\
&= \int_a^b |w(t)| dt.
\end{aligned}$$

□