Lecture 20: Trigonometric Functions

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20.1 Defining sine and cosine

Recall that if $x \in \mathbb{R}$, then

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

Note what happens if we (somewhat blindly) let $x = i\theta$:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

This is the motivation for our earlier definition of $e^{i\theta}$. It now follows that for any $x \in \mathbb{R}$, we have

$$e^{ix} = \cos(x) + i\sin(x) \text{ and } e^{-ix} = \cos(x) - i\sin(x),$$

from which we obtain (by addition)

$$2\cos(x) = e^{ix} + e^{-ix}$$

and (by subtraction)

$$2i\sin(x) = e^{ix} - e^{ix}.$$

Hence we have

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

and

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2},$$

which motivate the following definitions.

Definition 20.1. For any complex number z, we define the *sine* function by

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

and the *cosine* function by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

The following proposition is immediate from the properties of analytic functions and the fact that e^z is an entire function.

Proposition 20.1. Both $\sin(z)$ and $\cos(z)$ are entire functions.

Proposition 20.2. For all $z \in \mathbb{C}$,

$$\frac{d}{dz}\sin(z) = \cos(z) \text{ and } \frac{d}{dz}\cos(z) = -\sin(z).$$

Proof. We have

$$\frac{d}{dz}\sin(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos(z)$$

and

$$\frac{d}{dz}\cos(z) = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z).$$

□

20.2 Properties of sine and cosine

Proposition 20.3. For any $z \in \mathbb{C}$,

$$\sin(-z) = -\sin(z) \text{ and } \cos(-z) = \cos(z).$$

Proof. We have

$$\sin(-z) = \frac{e^{-iz} - e^{iz}}{2i} = -\sin(z)$$

and

$$\cos(-z) = \frac{e^{-iz} + e^{iz}}{2} = \cos(z).$$

□

Proposition 20.4. For any $z_1, z_2 \in \mathbb{C}$,

$$2\sin(z_1)\cos(z_2) = \sin(z_1 + z_2) + \sin(z_1 - z_2).$$

Proof. We have

$$\begin{aligned}
2\sin(z_1)\cos(z_2) &= 2\left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) \\
&= \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)}}{2i} \\
&= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} + \frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{2i} \\
&= \sin(z_1 + z_2) + \sin(z_1 - z_2).
\end{aligned}$$

□

Proposition 20.5. For any $z_1, z_1 \in \mathbb{C}$,

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$$

and

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2).$$

Proof. From the previous result, we have

$$2\sin(z_1)\cos(z_2) = \sin(z_1 + z_2) + \sin(z_1 - z_2)$$

and

$$2\sin(z_2)\cos(z_1) = \sin(z_1 + z_2) - \sin(z_1 - z_2),$$

from which we obtain the first identify by addition. It now follows that if

$$f(z) = \sin(z + z_2),$$

then

$$f(z) = \sin(z)\cos(z_2) + \cos(z)\sin(z_2)$$

as well. Hence

$$\cos(z_1 + z_2) = f'(z_1) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2).$$

$\square$

The following identities follow immediately from the previous propositions.

Proposition 20.6. For any $z \in \mathbb{C}$,

$$\sin^2(z) + \cos^2(z) = 1,$$

$$\sin(2z) = 2\sin(z)\cos(z),$$

$$\cos(2z) = 2\cos^2(z) - \sin^2(z),$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos(z),$$

$$\sin\left(z - \frac{\pi}{2}\right) = -\cos(z),$$

$$\cos\left(z + \frac{\pi}{2}\right) = -\sin(z),$$

$$\cos\left(z - \frac{\pi}{2}\right) = \sin(z),$$

$$\sin(z + \pi) = -\sin(z),$$

$$\cos(z + \pi) = -\cos(z),$$

$$\sin(z + 2\pi) = \sin(z),$$

and

$$\cos(z + 2\pi) = \cos(z).$$

Proposition 20.7. For any $z = x + iy \in \mathbb{C}$,

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

and

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

Proof. We first note that

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh(y)$$

and

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i\frac{e^{y} - e^{-y}}{2} = i\sinh(y).$$

Hence

$$\sin(x+iy) = \sin(x)\cos(iy) + \sin(iy)\cos(x) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

and

$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

□

It now follows (see the homework)that

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$$

and

$$|\cos(z)|^2 = \cos^2(x) + \sinh^2(y).$$

Since $\sinh(y) = 0$ if and only if $y = 0$, we see that $\sin(z) = 0$ if and only if $y = 0$ and $x = n\pi$ for some $n = 0, \pm 1, \pm 2, \ldots$, and $\cos(z) = 0$ if and only if $y = 0$ and $x = \frac{\pi}{2} + n\pi$ for some $n = 0, \pm 1, \pm 2, \ldots$. That is, $\sin(z) = 0$ if and only if

$$z = n\pi, n = 0, \pm 1, \pm 2, \ldots,$$

and $\cos(z) = 0$ if and only if

$$z = \frac{\pi}{2} + n\pi, n = 0, \pm 1, \pm 2, \ldots.$$

20.3 The other trigonometric functions

The rest of the trigonometric functions are defined as usual:

$$\tan(z) = \frac{\sin(z)}{\cos(z)},$$

$$\cot(z) = \frac{\cos(z)}{\sin(z)},$$

$$\sec(z) = \frac{1}{\cos(z)},$$

and

$$\csc(z) = \frac{1}{\sin(z)}.$$

Using our results on derivatives, it is straightforward to show that

$$\frac{d}{dz}\tan(z) = \sec^2(z),$$

$$\frac{d}{dz}\cot(z) = -\csc^2(z),$$

$$\frac{d}{dz}\sec(z) = \sec(z)\tan(z),$$

and

$$\frac{d}{dz}\csc(z) = -\csc(z)\cot(z).$$

In particular, these functions are analytic at all points at which they are defined. As with their real counterparts, they are all periodic, $\tan(z)$ and $\cot(z)$ having period π and $\sec(z)$ and $\csc(z)$ having period 2π.