Lecture 19: Complex Exponents

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19.1 Complex exponents

Definition 19.1. Given $c \in \mathbb{C}$ and $z \in \mathbb{C}$, $z \neq 0$, we define

$$z^c = e^{c \log(z)}.$$

Example 19.1. We have

$$2^i = e^{i \log(2)} = e^{i(\ln(2) + i2n\pi)} = e^{i \ln(2) - 2n\pi} = e^{-2n\pi}(\cos(\ln(2)) + i \sin(\ln(2))),$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Example 19.2. We have

$$i^i = e^{i \log(i)} = e^{i\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right)} = e^{-\left(2n + \frac{1}{2}\right)\pi}, n = 0, \pm 1, \pm 2, \ldots.$$

Let $\alpha \in \mathbb{R}$. If, for $z = re^{i\theta}$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$, we choose the branch

$$\log(z) = \ln(r) + i\theta$$

of $\log(z)$, we obtain a branch

$$z^c = e^{c \log(z)}$$

of z^c. We then have

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \log(z)} = e^{c \log(z)} \frac{c}{z} = \frac{ce^{c \log(z)}}{e^{\log(z)}} = ce^{(c-1)\log(z)} = cz^{c-1}$$

for z in
$$U = \{z = re^{i\theta} \in \mathbb{C} : r > 0, \alpha < \theta < \theta + 2\pi\}.$$
In particular, z^c is an analytic function of z for $z \in U$.

The principal value of z^c is
$$\text{P.V.}\, z^c = e^{c\text{Log}\,(z)}.$$
This is the principal branch of z^c, and is analytic in
$$\{z = re^{i\theta} \in \mathbb{C} : r > 0, -\pi < \theta < \pi\}.$$

Example 19.3. P.V. $i^i = e^{-\frac{\pi}{2}}$.

Example 19.4. The principle branch of $z^{\frac{3}{5}}$ is
$$\text{P.V.}\, z^{\frac{3}{5}} = e^{\frac{3}{5}\text{Log}\,(z)} = e^{\frac{3}{5}(\ln(r)+i\Theta)} = \sqrt[5]{r^3}\left(\cos\left(\frac{3\Theta}{5}\right) + i\sin\left(\frac{3\Theta}{5}\right)\right),$$
where $z = r^{i\Theta}$, $r > 0$ and $\Theta = \text{Arg}\,(z)$.

For a fixed $c \in \mathbb{C}$, $c \neq 0$, fix a value of $\log(c)$ and consider the function c^z. Then
$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z\log(c)} = e^{z\log(c)}\log(c) = c^z\log(c).$$
Hence c^z is an entire function of z. Moreover, note that if $c = e$ and we use $\text{Log}\,(e)$, this reduces to our previous result about e^z.