Lecture 18: Properties of Logarithms

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18.1 Branches

Note that

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg} z$$

is not continuous for any $z_0 = x_0 + iy_0$ with $y_0 = 0$ and $x_0 \leq 0$ since $\operatorname{Log}(z) \to \ln|x_0| + i\pi$ as $z = x + iy$ approaches z_0 with $y > 0$ and $\operatorname{Log}(z) \to \ln|x_0| - i\pi$ as $z = x + iy$ approaches z_0 with $y < 0$. However, if we restrict to $z = re^{i\theta}$ with $-\pi < \theta < \pi$ and write $\operatorname{Log}(z) = u(r, \theta) + iv(r, \theta)$, then

$$u(r, \theta) = \ln(r) \text{ and } v(r, \theta) = \theta,$$

and so

$$u_r(r, \theta) = \frac{1}{r} \text{ and } u_\theta(r, \theta) = 0$$

and

$$v_r(r, \theta) = 0 \text{ and } v_\theta(r, \theta) = 1.$$

Hence

$$ru_r(r, \theta) = v_\theta(r, \theta) \text{ and } u_\theta(r, \theta) = -rv_r(r, \theta).$$

That is, u and v satisfy the Cauchy-Riemann equations, and so $\operatorname{Log}(z)$ is analytic in

$$U = \{z = re^{i\theta} \in \mathbb{C} : r > 0, -\pi < \theta < \pi\}.$$

Moroever, for all $z \in U$,

$$\frac{d}{dz}\operatorname{Log} z = e^{-i\theta}(u_r(r,\theta) + iv_r(r,\theta) = e^{-i\theta}\left(\frac{1}{r} + i\cdot 0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

More generally, if for any real number α we restrict $\log(z)$ to

$$\log z = \ln r + i\theta,$$

where $z = re^{i\theta}$, $r > 0$, and $\alpha < \theta < \alpha + 2\pi$, then $\log z$ is analytic in

$$U = \{z = re^{i\theta} \in \mathbb{C} : r > 0, \alpha < \theta < \alpha + 2\pi\}$$

with

$$\frac{d}{dz}\log z = \frac{1}{z}.$$

We call such a restricted version of $\log z$ a *branch* of the multi-valued function $\log z$, with the restricted version of $\operatorname{Log} z$ discussed above being the *principal branch*. We call the origin along with the ray consisting of all points $z = re^{i\theta}$ for which $\theta = \alpha$ a *branch cut*; we call the origin a *branch point* because it is common to all the branch cuts.

18.2 Properties of logarithms

Proposition 18.1. For any $z_1, z_2 \in \mathbb{C}$, with $z_1 \neq 0$ and $z_2 \neq 0$, then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

and

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2).$$

Proof. We have

$$\begin{aligned}\log(z_1 z_2) &= \ln(|z_1 z_2|) + i\arg(z_1 z_2)\\ &= \ln(|z_1||z_2|) + i(\arg(z_1) + \arg(z_2))\\ &= (\ln(|z_1) + i\arg(z_1)) + (\ln(|z_2| + i\arg(z_2))\\ &= \log(z_1) + \log(z_2).\end{aligned}$$

and

$$\log\left(\frac{z_1}{z_2}\right) = \ln\left|\frac{z_1}{z_2}\right| + i\arg\left(\frac{z_1}{z_2}\right)$$

$$= \ln\left(\frac{|z_1|}{|z_2|}\right) + i(\arg(z_1) - \arg(z_2))$$
$$= (\ln(|z_1) + i\arg(z_1)) - (\ln(|z_2| + i\arg(z_2))$$
$$= \log(z_1) - \log(z_2).$$

□

Example 18.1. Let $z_1 = -2i$ and $z_2 = -i$. Then

$$\log(z_1) = \ln(2) + i\left(-\frac{\pi}{2} + 2n\pi\right), n = 0, \pm 1, \pm 2, \ldots,$$

$$\log(z_2) = i\left(-\frac{\pi}{2} + 2n\pi\right), n = 0, \pm 1, \pm 2, \ldots,$$

and

$$\log(z_1 z_2) = \log(-2) = \ln(2) + i(\pi + 2n\pi), n = 0, \pm 1, \pm 2, \ldots$$

Clearly,

$$\log(z_1 z_2) = \log(z_1) + \log(z_2).$$

However,

$$\operatorname{Log}(z_1) = \ln(2) - i\frac{\pi}{2},$$

$$\operatorname{Log}(z_2) = -i\frac{\pi}{2},$$

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(-2) = \ln(2) + i\pi,$$

and so

$$\operatorname{Log}(z_1) + \operatorname{Log}(z_2) = \ln(2) - i\pi \neq \operatorname{Log}(z_1 z_2).$$

As a prelude to discussing complex exponents, we note two more properties of logarithms. First, if $z = re^{i\theta}$, $r > 0$, then, since $z = e^{\log(z)}$, we have

$$z^n = e^{n\log(z)}, n = 0, \pm 1, \pm 2, \ldots.$$

Next, if n is a positive integer, $\Theta = \operatorname{Arg}(z)$, $z = re^{i\Theta} \neq 0$, then, for $k = 0, \pm 1, \pm 2, \ldots$,

$$e^{\frac{1}{n}\log(z)} = e^{\left(\frac{1}{n}\ln(r) + i\frac{\Theta + 2k\pi}{n}\right)} = \sqrt[n]{r}e^{i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)} = z^{\frac{1}{n}}.$$

This works as well when n is a negative integer by noting that

$$z^{\frac{1}{n}} = \left(z^{\frac{1}{-n}}\right)^{-1} = \left(e^{-\frac{1}{n}\log(z)}\right)^{-1} = e^{\frac{1}{n}\log(z)}.$$