Lecture 14: Cauchy-Riemann Equations: Polar Form

Dan Sloughter
Furman University
Mathematics 39

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14.1 Polar form of the Cauchy-Riemann Equations

Theorem 14.1. Suppose f is defined on an ϵ neighborhood U of a point $z_0 = r_0 e^{i\theta_0}$,

$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta),$$

and u_r, u_θ, v_r, and v_θ exist on U and are continuous at (r_0, θ_0). If f is differentiable at z_0, then

$$r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \text{ and } u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0)$$

and

$$f'(z_0) = e^{-i\theta_0}(u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)).$$

Proof. By the chain rule from multi-variable calculus,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}.$$

Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$u_r = u_x \cos(\theta) + u_y \sin(\theta)$$

and

$$u_\theta = -u_x r\sin(\theta) + u_y r\cos(\theta).$$

Similarly,

$$v_r = v_x\cos(\theta) + v_y\sin(\theta)$$

and

$$v_\theta = -v_x r\sin(\theta) + v_y r\cos(\theta).$$

Now f is differentiable at z_0, and so satisfies the Cauchy-Riemann equations at z_0. That is,

$$u_x(r_0,\theta_0) = v_y(r_0,\theta_0) \text{ and } u_y(r_0,\theta_0) = -v_x(r_0,\theta_0).$$

Hence

$$v_r(r_0,\theta_0) = -u_y(r_0,\theta_0)\cos(\theta) + u_x(r_0,\theta_0)\sin(\theta)$$

and

$$v_\theta(r_0,\theta_0) = u_y(r_0,\theta_0)r\sin(\theta) + u_x(r_0,\theta_0)r\cos(\theta).$$

It follows that

$$r_0u_r(r_0,\theta_0) = v_\theta(r_0,\theta_0) \text{ and } u_\theta(r_0,\theta_0) = -r_0v_r(r_0,\theta_0)$$

The final statement of the theorem is left as an exercise. □

Theorem 14.2. Suppose f is defined on an ϵ neighborhood U of a point $z_0 = r_0e^{i\theta_0}$,

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta),$$

and u_r, u_θ, v_r, and v_θ exist on U and are continuous at (r_0,θ_0). If

$$r_0u_r(r_0,\theta_0) = v_\theta(r_0,\theta_0) \text{ and } u_\theta(r_0,\theta_0) = -r_0v_r(r_0,\theta_0),$$

then f is differentiable at z_0.

Proof. The proof is left as a homework exercise. □

Example 14.1. For $z \neq 0$, let

$$f(z) = \frac{1}{z^2}.$$

If we write $z = re^{i\theta}$, then

$$f(z) = \frac{1}{r^2e^{2i\theta}} = \frac{1}{r^2}(\cos(2\theta) - i\sin(2\theta)).$$

Hence, in the notation of the above theorems,

$$u(r,\theta) = \frac{1}{r^2}\cos(2\theta)$$

and

$$v(r,\theta) = -\frac{1}{r^2}\sin(2\theta).$$

It follows that

$$u_r(r,\theta) = -\frac{2}{r^3}\cos(2\theta) \text{ and } u_\theta(r,\theta) = -\frac{2}{r^2}\sin(2\theta)$$

and

$$v_r(r,\theta) = \frac{2}{r^3}\sin(2\theta) \text{ and } v_\theta(r,\theta) = -\frac{2}{r^2}\cos(2\theta).$$

Thus

$$ru_r(r,\theta) = v_\theta(r,\theta) \text{ and } u_\theta(r,\theta) = -rv_r(r,\theta),$$

and so f is differentiable at all $z \neq 0$. Moreover,

$$\begin{aligned} f'(z) &= e^{-i\theta}\left(-\frac{2}{r^3}\cos(2\theta) + i\frac{2}{r^3}\sin(2\theta)\right) \\ &= -\frac{2}{r^3}e^{-i\theta}e^{-2i\theta} \\ &= -\frac{2}{r^3}e^{-3i\theta} \\ &= -\frac{2}{z^3}. \end{aligned}$$