14.1 Polar form of the Cauchy-Riemann Equations

Theorem 14.1. Suppose \( f \) is defined on an \( \epsilon \) neighborhood \( U \) of a point \( z_0 = r_0e^{i\theta_0} \),
\[
f(re^{i\theta}) = u(r, \theta) + iv(r, \theta),
\]
and \( u_r, u_\theta, v_r, \) and \( v_\theta \) exist on \( U \) and are continuous at \( (r_0, \theta_0) \). If \( f \) is differentiable at \( z_0 \), then
\[
r_0u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \quad \text{and} \quad u_\theta(r_0, \theta_0) = -r_0v_r(r_0, \theta_0)
\]
and
\[
f'(z_0) = e^{-i\theta_0}(u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)).
\]

Proof. By the chain rule from multi-variable calculus,
\[
\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}
\]
and
\[
\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta}
\]

Since \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \), we have
\[
u_r = u_x \cos(\theta) + u_y \sin(\theta)
\]
and
\[ u_\theta = -u_x r \sin(\theta) + u_y r \cos(\theta). \]

Similarly,
\[ v_r = v_x \cos(\theta) + v_y \sin(\theta) \]
and
\[ v_\theta = -v_x r \sin(\theta) + v_y r \cos(\theta). \]

Now \( f \) is differentiable at \( z_0 \), and so satisfies the Cauchy-Riemann equations at \( z_0 \). That is,
\[ u_x(r_0, \theta_0) = v_y(r_0, \theta_0) \quad \text{and} \quad u_y(r_0, \theta_0) = -v_x(r_0, \theta_0). \]

Hence
\[ v_r(r_0, \theta_0) = -u_y(r_0, \theta_0) \cos(\theta) + u_x(r_0, \theta_0) \sin(\theta) \]
and
\[ v_\theta(r_0, \theta_0) = u_y(r_0, \theta_0) r \sin(\theta) + u_x(r_0, \theta_0) r \cos(\theta). \]

It follows that
\[ r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \quad \text{and} \quad u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0). \]

The final statement of the theorem is left as an exercise. \( \square \)

**Theorem 14.2.** Suppose \( f \) is defined on an \( \epsilon \) neighborhood \( U \) of a point \( z_0 = r_0 e^{i\theta_0} \),
\[ f(re^{i\theta}) = u(r, \theta) + iv(r, \theta), \]
and \( u_r, u_\theta, v_r, \) and \( v_\theta \) exist on \( U \) and are continuous at \( (r_0, \theta_0) \). If
\[ r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \quad \text{and} \quad u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0), \]
then \( f \) is differentiable at \( z_0 \).

**Proof.** The proof is left as a homework exercise. \( \square \)

**Example 14.1.** For \( z \neq 0 \), let
\[ f(z) = \frac{1}{z^2}. \]

If we write \( z = re^{i\theta} \), then
\[ f(z) = \frac{1}{r^2 e^{2i\theta}} = \frac{1}{r^2} (\cos(2\theta) - i \sin(2\theta)). \]
Hence, in the notation of the above theorems,

\[ u(r, \theta) = \frac{1}{r^2} \cos(2\theta) \]

and

\[ v(r, \theta) = -\frac{1}{r^2} \sin(2\theta). \]

It follows that

\[ u_r(r, \theta) = -\frac{2}{r^3} \cos(2\theta) \quad \text{and} \quad u_\theta(r, \theta) = -\frac{2}{r^2} \sin(2\theta) \]

and

\[ v_r(r, \theta) = \frac{2}{r^3} \sin(2\theta) \quad \text{and} \quad v_\theta(r, \theta) = -\frac{2}{r^2} \cos(2\theta). \]

Thus

\[ ru_r(r, \theta) = v_\theta(r, \theta) \quad \text{and} \quad u_\theta(r, \theta) = -rv_r(r, \theta), \]

and so \( f \) is differentiable at all \( z \neq 0 \). Moreover,

\[
f'(z) = e^{-i\theta} \left( -\frac{2}{r^3} \cos(2\theta) + i \frac{2}{r^3} \sin(2\theta) \right)
\]

\[ = -\frac{2}{r^3} e^{-i\theta} e^{-2i\theta} \]

\[ = -\frac{2}{r^3} e^{-3i\theta} \]

\[ = -\frac{2}{z^3}. \]