13.1 Necessity of the Cauchy-Riemann Equations

Suppose \( f \) is differentiable at a point \( z_0 \in \mathbb{C} \). If we write \( z = x + iy \), \( z_0 = x_0 + iy_0 \), \( \Delta z = \Delta x + i\Delta y \), and

\[
f(z) = f(x + iy) = u(x, y) + iv(x, y),
\]

then

\[
f(z_0 + \Delta z) - f(z_0) = (u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)) \\
- (u(x_0, y_0) + iv(x_0, y_0)) \\
= (u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)) \\
+ i(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)).
\]

Hence

\[
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z} \\
+ i \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta z}.
\]

If we let \( \Delta z \to 0 \) along the real axis, then \( \Delta z = \Delta x \), and we have

\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
\]
\[
\lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = u_x(x_0, y_0) + iv_x(x_0, y_0).
\]

If we let \(\Delta z \to 0\) along the imaginary axis, then \(\Delta z = i\Delta y\), and we have

\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}
= -iu_y(x_0, y_0) + v_y(x_0, y_0).
\]

It follows that

\[
u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).
\]

We call these equations the Cauchy-Riemann equations.

**Theorem 13.1.** Suppose \(f\) is differentiable at \(z_0 = x_0 + iy_0\) and

\[
f(x + iy) = u(x, y) + iv(x, y).
\]

Then

\[
u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0)
\]

and

\[
f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).
\]

**Example 13.1.** If \(f(z) = |z|^2\), then, writing \(f(x + iy) = u(x, y) + iv(x, y)\),

\[
u(x, y) = x^2 + y^2 \text{ and } v(x, y) = 0.
\]

Hence

\[
u_x(x, y) = 2x, u_y(x, y) = 2y, \text{ and } v_x(x, y) = 0 = v_y(x, y).
\]

Hence \(f\) satisfies the Cauchy-Riemann equations only at the origin, showing that, as we have seen before, \(f\) is not differentiable at any \(z \neq 0\).
13.2 Sufficiency of the Cauchy-Riemann equations

By themselves, the Cauchy-Riemann equations are not sufficient to guarantee
the differentiability of a given function. However, the additional assumption
of continuity of the partial derivatives does suffice to guarantee differentia-


Theorem 13.2. Suppose $f$ is defined on an $\epsilon$ neighborhood $U$ of a point
$z_0 = x_0 + y_0$, $f(x + iy) = u(x, y) + iv(x, y)$,
and $u_x, u_y, v_x, v_y$ exist on $U$ and are continuous at $(x_0, y_0)$. If
$u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$,
then $f$ is differentiable at $z_0$.

Proof. Let $\Delta z = \Delta x + i\Delta y$,
$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$,
and
$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$.

The key fact we need comes from a theorem of multi-variable calculus: under
the conditions of the theorem, $u$ and $v$ are both differentiable functions. This
means that there exist $\epsilon_1$ and $\epsilon_2$, depending on $\Delta x$ and $\Delta y$, such that

$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1|\Delta z|$

and

$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_2|\Delta z|$

and both $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta z \to 0$. It follows that

$f(z_0 + \Delta z) - f(z_0) = \Delta u + i\Delta v$

$= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1|\Delta z|$

$+ i(v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_2|\Delta z|)$

$= u_x(x_0, y_0)\Delta x - v_x(x_0, y_0)\Delta y + iv_x(x_0, y_0)\Delta x$

$+ iu_y(x_0, y_0)\Delta y + (\epsilon_1 + i\epsilon_2)|\Delta z|$

$= u_x(x_0, y_0)\Delta z + iv_x(x_0, y_0)\Delta z + (\epsilon_1 + i\epsilon_2)|\Delta z|$. 

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Hence
\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
\]
\[
= \lim_{\Delta z \to 0} \left( u_x(x_0, y_0) + iv_x(x_0, y_0) + (\epsilon_1 + i\epsilon_2) \frac{\Delta z}{z} \right)
\]
\[
= u_x(x_0, y_0) + iv_x(u_0, y_0),
\]
since
\[
\lim_{\Delta z \to 0} (\epsilon_1 + \epsilon_2) = 0
\]
and
\[
\left| \frac{|z|}{z} \right| = \frac{|z|}{|z|} = 1
\]
for all \( z \).

\[\square\]

**Example 13.2.** It follows now from the previous example that \( f(z) = |z|^2 \) is differentiable at \( z = 0 \) but not for any \( z \neq 0 \).

**Example 13.3.** If \( f(z) = e^z \), then
\[
f(x + iy) = e^x e^{iy} = e^x \cos(y) + i e^x \sin(y).
\]
Hence
\[
u(x, y) = e^x \cos(y) \text{ and } v(x, y) = e^x \sin(y).
\]
Thus
\[
u_x(x, y) = e^x \cos(y) = v_y \text{ and } u_y(x, y) = -e^x \sin(y) = -v_x(x, y).
\]
Hence \( f \) is differentiable at every \( z \in \mathbb{C} \). Moreover,
\[
f'(z) = u_x(x, y) + iv_x(x, y) = e^x \cos(y) + i e^x \sin(y) = e^z = f(z).
\]