Lecture 12: Derivatives

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Mathematics 39

March 24, 2004

12.1 The derivative

Definition 12.1. Suppose f is defined on a neighborhood of a point $z_0 \in \mathbb{C}$. We say f is *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, in which case we call

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

the *derivative* of f at z_0.

Note that, letting $\Delta z = z - z_0$, we could also write

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Moreover, if we let $w = f(z)$ and $\Delta w = f(z + \Delta z) - f(z)$, then we may write

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}.$$

Example 12.1. Suppose $f(z) = z^n$, where n is a positive integer. Then

$$\begin{aligned} f(z+\Delta z) - f(z) &= (z+\Delta z)^n - z^n \\ &= (z^n + nz^{n-1}\Delta z + \cdots + nz(\Delta z)^{n-1} + (\Delta z)^n) - z^n \\ &= nz^{n-1}\Delta z + \cdots + nz(\Delta z)^{n-1} + (\Delta z)^n, \end{aligned}$$

so

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = nz^{n-1} + \cdots + nz(\Delta z)^{n-2} + (\Delta z)^{n-1}.$$

Hence

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = nz^{n-1}.$$

Example 12.2. Let

$$f(z) = |z|^2 = z\bar{z}.$$

Then

$$\begin{aligned} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{(z+\Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} \\ &= \frac{z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z}}{\Delta z} \\ &= \bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}. \end{aligned}$$

It follows that

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} \to \bar{z} + z$$

as $\Delta z \to 0$ along the real axis and

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} \to \bar{z} - z$$

as $\Delta z \to 0$ along the imaginary axis. Hence f is not differentiable at any $z \neq 0$. If $z = 0$, then

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0,$$

and so $f'(0) = 0$. Note that if we write $f(x+iy) = u(x,y) + iv(x,y)$, then

$$u(x,y) = x^2 + y^2$$

and

$$v(x,y) = 0.$$

Hence u and v have continuous partial derivatives of all order. This shows that the differentiability of u and v does not imply that f is differentiable. Moreover, note that this also shows that a function may be continuous at a point without being differentiable at that point.

Proposition 12.1. If f is differentiable at z_0, then f is continuous at z_0.

Proof. We need to show that

$$\lim_{z \to z_0} f(z) = f(z_0),$$

or, equivalently, that

$$\lim_{z \to z_0} (f(z) - f(z_0)) = 0.$$

The latter follows from

$$\begin{aligned} \lim_{z \to z_0} (f(z) - f(z_0)) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) \\ &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) \\ &= f'(z_0)(0) \\ &= 0. \end{aligned}$$

□

12.2 Differentiation formulas

Proposition 12.2. If $c \in \mathbb{C}$ and $f(z) = c$ for all $c \in \mathbb{C}$, then $f'(z) = 0$ for all $z \in \mathbb{C}$.

Proof. We have

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \lim_{w \to z} \frac{c - c}{w - z} = 0.$$

□

Proposition 12.3. If $c \in \mathbb{C}$ and f is differentiable at z, then

$$\frac{d}{dz}(cf(z)) = cf'(z).$$

Proof. We have

$$\frac{d}{dz}(cf(z)) = \lim_{w \to z} \frac{cf(w) - cf(z)}{w - z} = c \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = cf'(z).$$

□

Proposition 12.4. If f and g are both differentiable at z, then

$$\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z),$$

$$\frac{d}{dz} f(z)g(z) = f(z)g'(z) + g(z)f'(z),$$

and, if $g(z) \neq 0$,

$$\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}.$$

Proof. For the first statement, we have

$$\begin{aligned}
\frac{d}{dz}(f(z) + g(z)) &= \lim_{w \to z} \frac{(f(w) + g(w)) - (f(z) + g(z))}{w - z} \\
&= \lim_{w \to z} \left(\frac{f(w) - f(z)}{w - z} + \frac{g(w) - g(z)}{w - z} \right) \\
&= f'(z) + g'(z).
\end{aligned}$$

For the second,

$$\begin{aligned}
\frac{d}{dz} f(z)g(z) &= \lim_{w \to z} \frac{f(w)g(w) - f(z)g(z)}{w - z} \\
&= \lim_{w \to z} \frac{f(w)g(w) - f(z)g(w) + f(z)g(w) - f(z)g(z)}{w - z} \\
&= \lim_{w \to z} \left(g(w)\frac{f(w) - f(z)}{w - z} + f(z)\frac{g(w) - g(z)}{w - z} \right) \\
&= g(z)f'(z) + f(z)g'(z).
\end{aligned}$$

And for the third,

$$
\begin{aligned}
\frac{d}{dz}\frac{f(z)}{g(z)} &= \lim_{w\to z}\frac{\dfrac{f(w)}{g(w)}-\dfrac{f(z)}{g(z)}}{w-z} \\
&= \lim_{w\to z}\frac{f(w)g(z)-f(z)g(w)}{g(w)g(z)(w-z)} \\
&= \lim_{w\to z}\frac{f(w)g(z)-f(z)g(z)+f(z)g(z)-f(z)g(w)}{g(w)g(z)(w-z)} \\
&= \lim_{w\to z}\frac{g(z)\dfrac{f(w)-f(z)}{w-z}-f(z)\dfrac{g(w)-g(z)}{w-z}}{g(w)(g(z))} \\
&= \frac{g(z)f'(z)-f(z)g'(z)}{(g(z))^2}.
\end{aligned}
$$

□

Proposition 12.5. If f is differentiable at z_0 and g is differentiable at $f(z_0)$, then

$$(g\circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Proof. Let $w_0 = f(z_0)$ and choose $\epsilon > 0$ so that g is defined on the ϵ neighborhood of w_0. Call this neighborhood W. For $w \in W$, define

$$\Phi(w) = \begin{cases} \dfrac{g(w)-g(w_0)}{w-w_0} - g'(w_0), & \text{if } w \neq w_0, \\ 0, & \text{if } w = w_0. \end{cases}$$

Note that

$$\lim_{w\to w_0}\Phi(w) = g'(w_0) - g'(w_0) = 0 = \Phi(w_0),$$

so Φ is continuous at w_0. It also follows that

$$g(w) - g(w_0) = (g'(w_0) + \Phi(w))(w - w_0)$$

for all $w \in W$. Now choose $\delta > 0$ so that f is defined for all z in the δ neighborhood of z_0 and $f(z) \in W$ whenever z is in this neighborhood (such a δ exists because f is continuous at z_0). Call this neighborhood U. We then have that

$$g(f(z)) - g(f(z_0)) = (g'(f(z_0)) + \Phi(f(z)))(f(z) - f(z_0))$$

for all $z \in U$. Hence we have

$$\begin{aligned}(g \circ f)'(z_0) &= \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} \\ &= \lim_{z \to z_0} (g'(f(z_0)) + \Phi(f(z))) \frac{f(z) - f(z_0)}{z - z_0} \\ &= g'(f(z_0)) f'(z_0).\end{aligned}$$

□