Simply connected domains

- We say a domain $D$ is *simply connected* if, whenever $C \subset D$ is a simple closed contour, every point in the interior of $C$ lies in $D$.
- We say a domain which is not simply connected is *multiply connected*.

**Examples**

- The domain
  \[ U = \{ z \in \mathbb{C} : |z| < 1 \} \]
  is simply connected.
- The domain
  \[ A = \{ z \in \mathbb{C} : 1 < |z| < 2 \} \]
  is not simply connected.
Theorem

▶ If $D$ is a simply connected domain and $f$ is analytic in $D$, then

\[ \int_C f(z)dz = 0 \]

for every closed contour $C$ in $D$.

▶ Proof

▶ If $C$ is a simple closed contour, then the conclusion follows from the Cauchy-Goursat theorem.

▶ If $C$ is not simple, but intersects itself only a finite number of times, then the conclusion follows by writing $C$ as a sum of simple closed contours.

▶ We will omit the more difficult situation in which $C$ intersects itself an infinite number of times.

Corollary

▶ If $D$ is a simply connected domain and $f$ is analytic in $D$, then $f$ has an antiderivative at all points of $D$.

▶ Note: In particular, entire functions have antiderivatives on all of $\mathbb{C}$. 
Theorem

- Suppose $C$ is a positively oriented, simple closed contour and that $C_1$, $C_2$, \ldots $C_n$ are negatively oriented, simple closed contours, all of which are in the interior of $C$, are disjoint, and have disjoint interiors.
- Let $R$ be the region consisting of $C$, $C_1$, $C_2$, \ldots, $C_n$, and all points which are in the interior of $C$ and the exterior of each $C_k$.
- If $f$ is analytic in $R$, then
\[
\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.
\]

Proof

- Let $L_1$ be a polygonal path connecting $C$ to $C_1$, $L_k$ a polygonal path connecting $C_{k-1}$ to $C_k$, $k = 2, 3, \ldots, n$, and $L_{n+1}$ a polygonal path connecting $C_n$ to $C$.
- Let $B_1$ be the part of $C$ from where $L_{n+1}$ joins $C$ to where $L_1$ joins $C$, $B_2$ the remaining part of $C$, $\alpha_k$ the part of $C_k$ between where $L_k$ and $L_{k+1}$ join $C_k$, and $\beta_k$ the remaining part of $C_k$.
- Let
\[
\Gamma_1 = B_1 + L_1 + \alpha_1 + L_2 + \alpha_2 + \cdots + \alpha_n + L_{n+1}
\]
and
\[
\Gamma_2 = B_2 - L_{n+1} + \beta_n - L_n + \beta_{n-1} - \cdots + \beta_1 - L_1.
\]
Proof (cont’d)

Then, by the Cauchy-Goursat theorem,

\[ \int_{\Gamma_1} f(z)\,dz = 0 = \int_{\Gamma_2} f(z)\,dz. \]

Hence

\[ 0 = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = \int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz. \]

Corollary

Suppose \( C_1 \) and \( C_2 \) are positively oriented, simply closed contours with \( C_2 \) lying in the interior of \( C_1 \).

Let \( R \) be the region consisting of \( C_1, C_2 \), and the part of the interior of \( C_1 \) which is in the exterior of \( C_2 \).

If \( f \) is analytic in \( R \), then

\[ \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz. \]

Proof: From the previous theorem, we have

\[ \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0. \]
Example

- By a homework exercise, if \( C_0 \) is any positively oriented circle with center at the origin, then

\[
\int_{C_0} \frac{1}{z} \, dz = 2\pi i.
\]

- It now follows that if \( C \) is any positively oriented, simple closed contour with the origin in its interior, then

\[
\int_{C} \frac{1}{z} \, dz = 2\pi i.
\]