Mathematics 39: Lecture 19

Hyperbolic Functions

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Hyperbolic sine and cosine

- For any $z \in \mathbb{C}$, we define the *hyperbolic sine function* by

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

and the *hyperbolic cosine function* by

$$\cosh(z) = \frac{e^z + e^{-z}}{2}.$$

Proposition

- For any $z \in \mathbb{C}$,
$$\frac{d}{dz}\sinh(z) = \cosh(z)$$
and
$$\frac{d}{dz}\cosh(z) = \sinh(z).$$
- Proof:
  - $\frac{d}{dz}\sinh(z) = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{e^z + e^{-z}}{2} = \cosh(z)$
  - $\frac{d}{dz}\cosh(z) = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{e^z - e^{-z}}{2} = \sinh(z)$

Properties

- Note:
$$\cos(z) = \cosh(iz) \text{ and } \sin(z) = -i\sinh(iz)$$
- And
$$\cosh(z) = \cos(iz) \text{ and } \sinh(z) = -i\sin(iz).$$
- It follows that:
  - $\sinh(-z) = -i\sin(-iz) = i\sin(iz) = -\sinh(z)$,
  - $\cosh(-z) = \cos(-iz) = \cos(iz) = \cosh(z)$,
  - $$\cosh^2(z) - \sinh^2(z) = \cos^2(iz) - (-i\sin(iz))^2 = \cos^2(iz) + \sin^2(iz) = 1.$$

Properties (cont'd)

- Moreover,

$$\begin{aligned}\sinh(z_1+z_2) &= -i\sin(i(z_1+z_2)) \\ &= -i\sin(iz_1)\cos(iz_2) - i\cos(iz_1)\sin(iz_2) \\ &= \sinh(z_1)\cosh(z_2)+\cosh(z_1)\sinh(z_2),\end{aligned}$$

and

$$\begin{aligned}\cosh(z_1+z_2) &= \cos(i(z_1+z_2)) \\ &= \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2) \\ &= \cosh(z_1)\cosh(z_2)+\sinh(z_1)\sinh(z_2).\end{aligned}$$

Properties (cont'd)

- If $z = x + iy$, then $iz = -y + ix$, and we have

$$\begin{aligned}\sinh(z) &= -i\sin(iz) \\ &= -i(\sin(-y)\cos(ix)+\cos(-y)\sin(ix)) \\ &= \sinh(x)\cos(y) + i\cosh(x)\sin(y),\end{aligned}$$

$$\begin{aligned}\cosh(z) &= \cos(iz) \\ &= \cos(-y)\cos(ix) - \sin(-y)\sin(ix) \\ &= \cosh(x)\cos(y) + i\sinh(x)\sin(y),\end{aligned}$$

$$|\sinh(z)|^2 = |-i\sin(iz)|^2 = \sin^2(-y)+\sinh^2(x) = \sinh^2(x)+\sin^2(y)$$

and

$$|\cosh(z)|^2 = |\cos(iz)|^2 = \cos^2(-y)+\sinh^2(x) = \sinh^2(x)+\cos^2(y).$$

Properties (cont'd)

- Note: It follows that
  - both $\sinh(z)$ and $\cosh(z)$ are periodic with period $2\pi i$,
  - $\sinh(z) = 0$ if and only if $z = n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$,
  - $\cosh(z) = 0$ if and only if $z = i\left(\frac{\pi}{2} + n\pi\right)$, $n = 0, \pm 1, \pm 2, \ldots$.

More hyperbolic functions

- In analogy with the the circular trigonometric functions, we also define
  - $\tanh(z) = \dfrac{\sinh(z)}{\cosh(z)}$,
  - $\coth(z) = \dfrac{\cosh(z)}{\sinh(z)}$,
  - $\operatorname{sech}(z) = \dfrac{1}{\cosh(z)}$,
  - $\operatorname{csch}(z) = \dfrac{1}{\sinh(z)}$.

More hyperbolic function (cont'd)

- Note: The above are all analytic in their domains of definition.
- One may show, as in the real-variable case, that
  - $\frac{d}{dz}\tanh(z) = \operatorname{sech}^2(z)$,
  - $\frac{d}{dz}\coth(z) = -\operatorname{csch}^2(z)$,
  - $\frac{d}{dz}\operatorname{sech}(z) = -\operatorname{sech}(z)\tanh(z)$,
  - $\frac{d}{dz}\operatorname{csch}(z) = -\operatorname{csch}(z)\coth(z)$.