Lecture 9: Angles and Triangles

9.1 Angles

Definition Given distinct noncollinear points A, B, and C in a metric geometry, we call the set

$$\overrightarrow{BA} \cup \overrightarrow{BC}$$

an *angle*, which we denote $\angle ABC$.

Note: An angle is a set of points, not a number. We will introduce the measure of an angle later. The distinction is similar to the distinction between the line segment $\overline{AB}$, which is a set of points, and the length of the line segment AB, which is a real number.

Theorem The only extreme point of an angle $\angle ABC$ is the point B.

Proof Suppose B is a passing point of $\angle ABC$. Then $P - B - Q$ for some points $P, Q \in \angle ABC$. Suppose $P \in \overrightarrow{BA}$. If $Q \in \overrightarrow{BA}$, then B is a passing point of $\overrightarrow{BA}$, contradicting our previous result that B is an extreme point of $\overrightarrow{BA}$. Hence we must have $Q \in \overrightarrow{BC}$. But then $\overrightarrow{BA} = \overrightarrow{BP}$ and $\overrightarrow{BC} = \overrightarrow{BQ}$, and so

$$\overleftrightarrow{AB} = \overleftrightarrow{BP} = \overleftrightarrow{BQ} = \overleftrightarrow{BC},$$

contradicting the assumption that A, B, and C are noncollinear. Hence B is an extreme point of $\angle ABC$.

Clearly, if $P \in \angle ABC$, $P \neq B$, then P is a passing point of $\angle ABC$ since P is a passing point of either $\overrightarrow{BA}$ or $\overrightarrow{BC}$.

Theorem In a metric geometry, if $\angle ABC = \angle DEF$, then $B = E$.

Proof Follows immediately from the previous theorem.

Definition Given an angle $\angle ABC$ in a metric geometry, we call B the *vertex* of $\angle ABC$.

9.2 Triangles

Definition Given three noncollinear points A, B, and C in a metric geometry, we call the set

$$\overline{AB} \cup \overline{BC} \cup \overline{AC}$$

a *triangle*, which we denote $\triangle ABC$.

Theorem The only extreme points of a triangle $\triangle ABC$ are the points A, B, and C.

Proof Suppose A is a passing point of $\triangle ABC$. Then $P - A - Q$ for some points $P, Q \in \triangle ABC$. Now P and Q cannot both belong to $\angle BAC$, for then A would be a passing point of $\angle BAC$. If $P, Q \in \overline{BC}$, then,

$$A \in \overleftrightarrow{PQ} = \overleftrightarrow{BC},$$

contradicting the noncollinearity of A, B, and C. Hence one, and only, of P or Q must lie on $\overline{AB} \cup \overline{AC}$. Suppose $P \in \overline{AB}$, $Q \in \overline{BC}$, $Q \notin \overline{AB}$ and $Q \notin \overline{AC}$. Then

$$\overleftrightarrow{AB} = \overleftrightarrow{AP} = \overleftrightarrow{AQ},$$

so $Q \in \overleftrightarrow{AB}$. But $B - Q - C$, so $Q, B \in \overleftrightarrow{AB}$ and $Q, B \in \overleftrightarrow{BC}$, which would make $\overleftrightarrow{AB} = \overleftrightarrow{BC}$, again contradicting the noncollinearity of A, B, and C. So A is an extreme point of $\triangle ABC$. Similarly, B and C are extreme points of $\triangle ABC$. Clearly, all other points of $\triangle ABC$ are passing points of $\triangle ABC$.

Theorem If, in a metric geometry, $\triangle ABC = \triangle DEF$, then $\{A, B, C\} = \{D, E, F\}$.

Proof Follows immediately from the previous theorem.

Definition Given a triangle $\triangle ABC$ in a metric geometry, we call A, B, and C the *vertices* of $\triangle ABC$, and we call $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ the *edges* of $\triangle ABC$.