Lecture 8: Line Segments and Rays

8.1 Line Segments

Definition Given distinct points A and B in a metric geometry, we call the set

$$\overline{AB} = \{C : C \in \overleftrightarrow{AB} \text{ and } C = A, C = B, \text{ or } A - C - B\}$$

the *line segment* from A to B.

Example Consider points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ on the line ${}_cL_r$ in the Poincaré Plane with $x_1 < x_2$. Suppose $C = (x, y)$ with $A - C - B$. If $f : {}_cL_r \to \mathbb{R}$ is the ruler defined by

$$f(x, y) = \log\left(\frac{x - c + r}{y}\right)$$

and $t_1 = f(A)$, $t_2 = f(B)$, and $t = f(C)$, then $t_1 * t * t_2$. Now $x_1 = c + r\tanh(t_1)$, $x = c + r\tanh(t)$, and $x_2 = c + r\tanh(t_2)$, and the hyperbolic tangent function is increasing, so $x_1 * x * x_2$. Since $x_1 < x_2$, we must have

$$x_1 < x < x_2.$$

Hence

$$\overline{AB} \subset \{C : C \in {}_cL_r, C = (x, y), x_1 \le x \le x_2\}.$$

Now suppose $C \in {}_cL_r$, $C = (x, y)$, and $x_1 \le x \le x_2$. Since the function $g(t) = c + r\tanh(t)$ is increasing, g^{-1} is increasing. Since $t_1 = f(A) = g^{-1}(x_1)$, $t = f(C) = g^{-1}(x)$, and $t_2 = f(B) = g^{-1}(x_2)$, it follows that $t_1 \le t \le t_2$. Hence either $C = A$ (it $t = t_1$), $A - C - B$ (if $t_1 < t < t_2$), or $C = B$ (if $t = t_2$). Thus

$$\{C : C \in {}_cL_r, C = (x, y), x_1 \le x \le x_2\} \subset \overline{AB}.$$

Hence

$$\overline{AB} = \{C : C \in {}_cL_r, C = (x, y), x_1 \le x \le x_2\}.$$

Definition Let $\mathcal{S}$ be a set of points in a metric geometry. We call a point $B \in S$ a *passing point* of S if there exist points P and Q in S such that $P - B - Q$. We call a point which is not a passing point of S an *extreme point* of S.

Theorem Given distinct points A and B in a metric geometry, the extreme points of $\overline{AB}$ are A and B.

Proof Suppose A is a passing point of $\overline{AB}$. Then there exist points $P, Q \in \overline{AB}$ such that $P - A - Q$. Now one of the following must hold: $B - P - A - Q$, $B = P$, $P - B - A - Q$,

$P - A - B - Q$, $B = Q$, or $P - A - Q - B$. The first three of these imply that $B - A - Q$, and so $Q \notin \overline{AB}$, and the last three imply $P - A - B$, and so $P \notin \overline{AB}$. Either of these conclusions contradicts our assumptions about P and Q. Hence A must be an extreme point of $\overline{AB}$. A similar argument shows that B is an extreme point of $\overline{AB}$.

Finally, consider a point $P \in \overline{AB}$, $P \neq A$ and $P \neq B$.Then $A - P - B$, so P is a passing point of $\overline{AB}$. Hence A and B are the only extreme points of $\overline{AB}$.

Theorem If A, B, C, and D are points in a metric geometry with $\overline{AB} = \overline{CD}$, then $\{A, B\} = \{C, D\}$

Proof The sets $\{A, B\}$ and $\{C, D\}$ are both characterized as the set of extreme points of $\overline{AB}$.

Definition We call the points A and B the *endpoints*, or *vertices*, of the line segment $\overline{AB}$.

Definition The *length* of a line segment $\overline{AB}$ is AB.

8.2 Rays

Definition Given distinct points A and B in a metric geometry, the *ray* from A toward B is the set

$$\overrightarrow{AB} = \overline{AB} \cup \{C : C \in \overleftrightarrow{AB}, A - B - C\}.$$

Theorem If AB are points in the Euclidean Plane $\{\mathbb{R}^2, \mathcal{L}_2, d_E\}$, then

$$\overline{AB} = \{C : C \in \mathbb{R}^2, C = A + t(B - A), 0 \leq t \leq 1\}$$

and

$$\overrightarrow{AB} = \{C : C \in \mathbb{R}^2, C = A + t(B - A), t \geq 0\}.$$

Proof Homework

Theorem If A, B, and C are points in a metric geometry with $C \in \overrightarrow{AB}$, $C \neq A$, then $\overrightarrow{AB} = \overrightarrow{AC}$.

Proof Homework

Theorem If A, B, and C are points in a metric geometry with $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.

Proof Homework

Definition We call the point A the *vertex*, or *initial point*, of the ray $\overrightarrow{AB}$.

Theorem If A and B are distinct points in a metric geometry, then there exists a ruler $f : \overleftrightarrow{AB} \to \mathbb{R}$ such that

$$\overrightarrow{AB} = \{P : P \in \overleftrightarrow{AB}, f(P) \geq 0\}.$$

Proof Let $f : \overleftrightarrow{AB} \to \mathbb{R}$ be ruler with origin A and B positive. Suppose $Q \in \overleftrightarrow{AB}$ and $f(Q) \geq 0$. Then either $f(Q) = 0$, $0 < f(Q) < f(B)$, $f(Q) = f(B)$, or $0 < f(B) < f(Q)$. Hence either $Q = A$, $A - Q - B$, $A = B$, or $A - B - Q$. Thus $Q \in \overrightarrow{AB}$, and so

$$\{P : P \in \overleftrightarrow{AB}, f(P) \geq 0\} \subset \overrightarrow{AB}.$$

Now suppose $Q \in \overrightarrow{AB}$. If $f(Q) < 0$, then $f(Q) < f(A) < f(B)$, so $Q - A - B$. But then $Q \notin \overrightarrow{AB}$. Hence we must have $f(Q) \geq 0$. Thus

$$\overrightarrow{AB} \subset \{P : P \in \overleftrightarrow{AB}, f(P) \geq 0\},$$

and so

$$\overrightarrow{AB} = \{P : P \in \overleftrightarrow{AB}, f(P) \geq 0\}.$$

8.3 Congruence

Definition We say line segments $\overline{AB}$ and $\overline{CD}$ are *congruent*, written $\overline{AB} \simeq \overline{CD}$, if $AB = CD$.

Segment Construction Given a ray $\overrightarrow{AB}$ and a segment $\overline{PQ}$ in a metric geometry, then there exists a unique point $C \in \overrightarrow{AB}$ with $\overline{PQ} \simeq \overline{AC}$.

Proof Let f be a ruler for $\overleftrightarrow{AB}$ with A as origin and B positive. Let $r = PQ$ and let $C = f^{-1}(r)$. Then, since $f(C) = r > 0$, $C \in \overrightarrow{AB}$ and

$$AC = |f(A) - f(C)| = |0 - r| = r = PQ,$$

so $\overline{AC} \simeq \overline{PQ}$. Now suppose $D \in \overrightarrow{AB}$ with $\overline{AD} \simeq \overline{PQ}$. Then, since $f(D) > 0$,

$$f(C) = r = AD = |f(D) - f(A)| = |f(D)| = f(D),$$

implying that $D = C$ (since f is injective). Hence the point C is unique.

Segment Addition In a metric geometry, if $A - B - C$, $P - Q - R$, $\overline{AB} \simeq \overline{PQ}$, and $\overline{BC} \simeq \overline{QR}$, then $\overline{AC} \simeq \overline{PR}$.

Proof Homework

Segment Subtraction In a metric geometry, if $A - B - C$, $P - Q - R$, $\overline{AB} \simeq \overline{PQ}$, and $\overline{AC} \simeq \overline{PR}$, then $\overline{BC} \simeq \overline{QR}$.

Proof Homework