8.1 Line Segments

Definition  Given distinct points $A$ and $B$ in a metric geometry, we call the set

$$\overline{AB} = \{ C : C \in \overline{AB} \text{ and } C = A, C = B, \text{ or } A - C - B \}$$

the line segment from $A$ to $B$.

Example  Consider points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ on the line $cL_r$ in the Poincaré Plane with $x_1 < x_2$. Suppose $C = (x, y)$ with $A - C - B$. If $f : cL_r \to \mathbb{R}$ is the ruler defined by

$$f(x, y) = \log \left( \frac{x - c + r}{y} \right)$$

and $t_1 = f(A)$, $t_2 = f(B)$, and $t = f(C)$, then $t_1 \ast t \ast t_2$. Now $x_1 = c + r \tanh(t_1)$, $x = c + r \tanh(t)$, and $x_2 = c + r \tanh(t_2)$, and the hyperbolic tangent function is increasing, so $x_1 \ast x \ast x_2$. Since $x_1 < x_2$, we must have

$$x_1 < x < x_2.$$

Hence

$$\overline{AB} \subset \{ C : C \in cL_r, C = (x, y), x_1 \leq x \leq x_2 \}.$$

Now suppose $C \in cL_r$, $C = (x, y)$, and $x_1 \leq x \leq x_2$. Since the function $g(t) = c + r \tanh(t)$ is increasing, $g^{-1}$ is increasing. Since $t_1 = f(A) = g^{-1}(x_1)$, $t = f(C) = g^{-1}(x)$, and $t_2 = f(B) = g^{-1}(x_2)$, it follows that $t_1 \leq t \leq t_2$. Hence either $C = A$ (it $t = t_1$), $A - C - B$ (if $t_1 < t < t_2$), or $C = B$ (if $t = t_2$). Thus

$$\{ C : C \in cL_r, C = (x, y), x_1 \leq x \leq x_2 \} \subset \overline{AB}.$$

Hence

$$\overline{AB} = \{ C : C \in cL_r, C = (x, y), x_1 \leq x \leq x_2 \}.$$

Definition  Let $S$ be a set of points in a metric geometry. We call a point $B \in S$ a passing point of $S$ if there exist points $P$ and $Q$ in $S$ such that $P - B - Q$. We call a point which is not a passing point of $S$ an extreme point of $S$.

Theorem  Given distinct points $A$ and $B$ in a metric geometry, the extreme points of $\overline{AB}$ are $A$ and $B$.

Proof  Suppose $A$ is a passing point of $\overline{AB}$. Then there exist points $P, Q \in \overline{AB}$ such that $P - A - Q$. Now one of the following must hold: $B - P - A - Q$, $B = P$, $P - B - A - Q$, etc.
$P - A - B - Q, B = Q$, or $P - A - Q - B$. The first three of these imply that $B - A - Q$, and so $Q \notin \overline{AB}$, and the last three imply $P - A - B$, and so $P \notin \overline{AB}$. Either of these conclusions contradicts our assumptions about $P$ and $Q$. Hence $A$ must be an extreme point of $\overline{AB}$. A similar argument shows that $B$ is an extreme point of $\overline{AB}$.

Finally, consider a point $P \in \overline{AB}$, $P \neq A$ and $P \neq B$. Then $A - P - B$, so $P$ is a passing point of $\overline{AB}$. Hence $A$ and $B$ are the only extreme points of $\overline{AB}$.

**Theorem** If $A$, $B$, $C$, and $D$ are points in a metric geometry with $\overline{AB} = \overline{CD}$, then $\{A, B\} = \{C, D\}$

**Proof** The sets $\{A, B\}$ and $\{C, D\}$ are both characterized as the set of extreme points of $\overline{AB}$.

**Definition** We call the points $A$ and $B$ the endpoints, or vertices, of the line segment $\overline{AB}$.

**Definition** The length of a line segment $\overline{AB}$ is $AB$.

### 8.2 Rays

**Definition** Given distinct points $A$ and $B$ in a metric geometry, the ray from $A$ toward $B$ is the set

$$\overrightarrow{AB} = \overline{AB} \cup \{C : C \in \overline{AB}, A - B - C\}.$$  

**Theorem** If $AB$ are points in the Euclidean Plane $\{\mathbb{R}^2, \mathcal{L}_2, d_E\}$, then

$$\overline{AB} = \{C : C \in \mathbb{R}^2, C = A + t(B - A), 0 \leq t \leq 1\}$$

and

$$\overrightarrow{AB} = \{C : C \in \mathbb{R}^2, C = A + t(B - A), t \geq 0\}.$$  

**Proof** Homework

**Theorem** If $A$, $B$, and $C$ are points in a metric geometry with $C \in \overline{AB}, C \neq A$, then $\overrightarrow{AB} = \overrightarrow{AC}$.

**Proof** Homework

**Theorem** If $A$, $B$, and $C$ are points in a metric geometry with $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.

**Proof** Homework
Definition  We call the point $A$ the vertex, or initial point, of the ray $\overrightarrow{AB}$.

Theorem  If $A$ and $B$ are distinct points in a metric geometry, then there exists a ruler $f : \overrightarrow{AB} \to \mathbb{R}$ such that

$$\overrightarrow{AB} = \{ P : P \in \overrightarrow{AB}, f(P) \geq 0 \}.$$

Proof  Let $f : \overrightarrow{AB} \to \mathbb{R}$ be ruler with origin $A$ and $B$ positive. Suppose $Q \in \overrightarrow{AB}$ and $f(Q) \geq 0$. Then either $f(Q) = 0$, $0 < f(Q) < f(B)$, $f(Q) = f(B)$, or $0 < f(B) < f(Q)$. Hence either $Q = A$, $A - Q - B$, $A = B$, or $A - B - Q$. Thus $Q \in \overrightarrow{AB}$, and so

$$\{ P : P \in \overrightarrow{AB}, f(P) \geq 0 \} \subset \overrightarrow{AB}.$$

Now suppose $Q \in \overrightarrow{AB}$. If $f(Q) < 0$, then $f(Q) < f(A) < f(B)$, so $Q - A - B$. But then $Q \notin \overrightarrow{AB}$. Hence we must have $f(Q) \geq 0$. Thus

$$\overrightarrow{AB} \subset \{ P : P \in \overrightarrow{AB}, f(P) \geq 0 \},$$

and so

$$\overrightarrow{AB} = \{ P : P \in \overrightarrow{AB}, f(P) \geq 0 \}.$$

8.3 Congruence

Definition  We say line segments $\overline{AB}$ and $\overline{CD}$ are congruent, written $\overline{AB} \simeq \overline{CD}$, if $\overline{AB} = \overline{CD}$.

Segment Construction  Given a ray $\overrightarrow{AB}$ and a segment $\overline{PQ}$ in a metric geometry, then there exists a unique point $C \in \overrightarrow{AB}$ with $\overline{PQ} \simeq \overline{AC}$.

Proof  Let $f$ be a ruler for $\overrightarrow{AB}$ with $A$ as origin and $B$ positive. Let $r = \overline{PQ}$ and let $C = f^{-1}(r)$. Then, since $f(C) = r > 0$, $C \in \overrightarrow{AB}$ and

$$AC = |f(A) - f(C)| = |0 - r| = r = \overline{PQ},$$

so $\overline{AC} \simeq \overline{PQ}$. Now suppose $D \in \overrightarrow{AB}$ with $\overline{AD} \simeq \overline{PQ}$. Then, since $f(D) > 0$, $f(C) = r = AD = |f(D) - f(A)| = |f(D)| = f(D),$ implying that $D = C$ (since $f$ is injective). Hence the point $C$ is unique.
**Segment Addition**  In a metric geometry, if $A - B - C$, $P - Q - R$, $\overline{AB} \simeq \overline{PQ}$, and $\overline{BC} \simeq \overline{QR}$, then $\overline{AC} \simeq \overline{PR}$.

**Proof**  Homework

**Segment Subtraction**  In a metric geometry, if $A - B - C$, $P - Q - R$, $\overline{AB} \simeq \overline{PQ}$, and $\overline{AC} \simeq \overline{PR}$, then $\overline{BC} \simeq \overline{QR}$.

**Proof**  Homework