Lecture 7: Betweenness

7.1 Betweenness

Definition If A, B, C are distinct, collinear points in a metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$ with

$$d(A, B) + d(B, C) = d(A, C),$$

then we say B is *between* A and C.

Notation: We write $A - B - C$ to denote that B is between A and C. Moreover, we will let AB denote $d(A, B)$ as long as the distance function d is clear from the context. In particular, we can say that, for distinct collinear points A, B, C in a metric geometry,

$$A - B - C \text{ if and only if } AB + BC = AC.$$

Example Let $A = (4, 4)$, $B = (1, 5)$, and $C = (5, 3)$ in the Poincaré plane. Then $\overleftrightarrow{AB} = {}_cL_r$, where

$$c = \frac{(16 - 25) + (16 - 1)}{2(4 - 1)} = \frac{6}{6} = 1$$

and

$$r = \sqrt{(4 - 1)^2 + 16} = 5.$$

Now C also lies on ${}_1L_5$ since

$$(5 - 1)^2 + 3^2 = 25.$$

Hence A, B, and C are collinear. Now $f : {}_1L_5 \to \mathbb{R}$ defined by

$$f(x, y) = \log\left(\frac{x - 1 + 5}{y}\right) = \log\left(\frac{x + 4}{y}\right)$$

is a ruler for ${}_1L_5$. The coordinates of A, B, and C are

$$f(A) = \log(2),$$

$$f(B) = \log(1) = 0,$$

and

$$f(C) = \log(3),$$

so

$$AB = |\log(2) - 0| = \log(2),$$

$$BC = |0 - \log(3)| = \log(3),$$

and

$$AC = |\log(2) - \log(3)| = \log\left(\frac{3}{2}\right).$$

Hence $BC = BA + AC$, so $B - A - C$.

Theorem If $A - B - C$, then $C - B - A$.

Proof Since $A - B - C$, A, B, and C are distinct and collinear. Moreover,

$$CA = AC = AB + BC = CB + BA.$$

Hence $C - B - A$.

7.2 Betweenness and rulers

Definition For real numbers x, y, and z, we say y is *between* x and z, denoted $x * y * z$, if either $x < y < z$ or $z < y < x$.

Theorem Suppose A, B, and C are three points on a line ℓ with ruler f. Let $x = f(A)$, $y = f(B)$, and $z = f(C)$. Then $A - B - C$ if and only if $x * y * z$.

Proof First note that since f is a bijection, A, B, and C are distinct if and only if x, y, and z are distinct.

Assume A, B, and C are distinct and suppose $A - B - C$. Then

$$AB = |x - y|,$$

$$BC = |y - z|,$$

and

$$AC = |x - z|,$$

so $AB + BC = AC$ implies that

$$|x - y| + |y - z| = |x - z|.$$

Now exactly one of the following is true: (1) $x < y < z$, (2) $z < y < z$, (3) $y < x < z$, (4) $z < x < y$, (5) $x < z < y$, or (6) $y < z < x$. Suppose (3) holds. Then

$$|x - y| = x - y,$$

$$|y - z| = z - y,$$

and

$$|x - z| = z - x.$$

Hence

$$(x - y) + (z - y) = z - x,$$

implying that

$$x = y,$$

a contradiction. Similarly, cases (4), (5), and (6) lead to contradictions. Hence either $x < y < z$ or $z < y < x$, in which case $x * y * z$.

Now suppose $x * y * z$. If $x < y < z$, then

$$|x - y| = y - x,$$

$$|y - z| = z - y,$$

and

$$|x - z| = z - x.$$

Hence

$$AB + BC = (y - x) + (z - y) = z - x = |x - z| = AC,$$

and so $A - B - C$. A similar argument holds if $z < y < x$.

Theorem Given three distinct points on a line in a metric geometry, one and only one of them is between the other two.

Proof Immediate consequence of the previous theorem and properties of the real numbers.

Theorem If A and B are distinct points in a metric geometry, then there exist points C and D with $A - B - C$ and $A - D - B$.

Proof Let f be a ruler for $\overleftrightarrow{AB}$ with $f(A) = 0$ and $f(B) > 0$. Let $y = f(B) + 1$ and let

$$z = \frac{f(B)}{2}.$$

Let $C = f^{-1}(y)$ and let $D = f^{-1}(z)$. Then $f(A) < f(B) < f(C)$, so $A - B - C$ and $f(A) < f(D) < f(B)$, so $A - D - B$.

Definition We define $A - B - C - D$ to mean $A - B - C$, $A - B - D$, $A - C - D$, and $B - C - D$.

Theorem $A - B - C - D$ if and only if $A - B - C$ and $B - C - D$.

Proof Homework