Lecture 6: The Cartesian Plane Revisited

6.1 Points as vectors

Definition If $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $\alpha \in \mathbb{R}$, then

$$A + B = (x_1 + x_2, y_1 + y_2),$$

$$\alpha A = (\alpha x_1, \alpha x_2),$$

$$A - B = A + (-1)B = (x_1 - x_2, y_1 - y_2),$$

$$\langle A, B \rangle = x_1 x_2 + y_1 y_2,$$

and

$$\|A\| = \sqrt{\langle A, A \rangle}.$$

Given any $A, B \in \mathbb{R}^2$, let

$$L_{AB} = \{P : P = A + t(B - A), t \in \mathbb{R}\}.$$

Then, in the Cartesian Plane $\{\mathbb{R}^2, \mathcal{L}_E\}$, $L_{AB} = \overleftrightarrow{AB}$ and, if we let

$$\mathcal{L} = \{L_{AB} : A, B \in \mathbb{R}^2\},$$

we have

$$\mathcal{L}_E = \mathcal{L}.$$

Recall: If $A, B \in \mathbb{R}^2$, then $d_E(A, B) = \|A - B\|$.

Theorem Given $\ell = L_{AB}$ in the Cartesian Plane, the function $f : \ell \to \mathbb{R}$ defined at $P = A + t(B - A)$ by

$$f(P) = t\|B - A\|$$

is a ruler for ℓ in $\{\mathbb{R}^2, \mathcal{L}_E, d_E\}$.

Proof Note that for any $P = A + t(B - A)$ and $Q = A + s(B - A)$, $t, s \in \mathbb{R}$, on ℓ, we have

$$d_E(P, Q) = \|A - B\| = \|t(B - A) - s(B - A)\| = \|B - A\||t - s|.$$

Hence

$$|f(P) - f(Q)\| = |t\|B - A\| - s\|B - A\|| = \|B - A\||t - s| = d_E(P, Q).$$

Thus f is a ruler for ℓ.

6.2 Two inequalities

Cauchy-Schwarz Inequality If $A, B \in \mathbb{R}^2$, then

$$|\langle A, B\rangle| \leq \|A\|\|B\|.$$

Moreover, equality holds if and only if either $B = (0,0)$ or $A = tB$ for some $t \in \mathbb{R}$.

Proof If $B = (0,0)$, then

$$|\langle A, B\rangle| = 0 = \|A\|\|B\|.$$

Suppose $B \neq (0,0)$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = \|A - tB\|^2.$$

Then

$$g(t) = \langle A - tB, A - tB\rangle = \langle A, A\rangle - 2t\langle A, B\rangle + t^2\langle B, B\rangle,$$

so $g(t)$ is a quadratic polynomial. Since $g(t) \geq 0$ for all t, it follows that g has at most one real zero. Hence, using the quadratic formula, we have

$$4\langle A, B\rangle^2 - 4\langle B, B\rangle\langle A, A\rangle \leq 0.$$

Hence

$$|\langle A, B\rangle| \leq \sqrt{\langle A, A\rangle\langle B, B\rangle} = \|A\|\|B\|.$$

Finally, note that

$$4\langle A, B\rangle^2 - 4\langle B, B\rangle\langle A, A\rangle = 0$$

if and only if $g(t)$ has a zero. In that case, there exists a $t \in \mathbb{R}$ such that

$$0 = g(t) = \|A - tB\|^2,$$

which is true if and only if $A = tB$.

Definition Suppose d is a distance function on a set $\mathcal{S}$. We say d satisfies the *triangle inequality* if, for all $A, B, C \in \mathcal{S}$,

$$d(A, C) \leq d(A, B) + d(B, C).$$

Theorem The Euclidean distance function, d_E, satisfies the triangle inequality.

Proof If $P, Q \in \mathbb{R}^2$, we have

$$\begin{aligned}
\|P + Q\|^2 &= \langle P + Q, P + Q\rangle \\
&= \langle P, P\rangle^2 + 2\langle P, Q\rangle + \langle Q, Q\rangle^2 \\
&= \|P\| + 2\langle P, Q\rangle + \|Q\| \\
&\leq \|P\| + 2|\langle P, Q\rangle| + \|Q\| \\
&\leq \|P\| + 2\|P\|\|Q\| + \|Q\| \\
&= (\|P\| + \|Q\|)^2,
\end{aligned}$$

from which it follows that

$$\|P+Q\| \leq \|P\| + \|Q\|.$$

If $A, B, C \in \mathbb{R}^2$, let $P = A - B$ and $Q = B - C$. Then

$$d_E(A,C) = \|A-C\| = \|(A-B)+(B-C)\| \leq \|A-B\|+\|B-C\| = d_E(A,B)+d_E(B,C).$$