Lecture 5: Coordinate Systems

5.1 Special rulers

Theorem If f is a ruler for a line ℓ in a metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$, then (1) $g : \ell \to \mathbb{R}$ defined by $g(P) = -f(P)$ and (2) $h : \ell \to \mathbb{R}$ defined by $h(P) = f(P) - a$, where $a \in \mathbb{R}$, are rulers for ℓ.

Proof If $t \in \mathbb{R}$, then there exists $P, Q \in \ell$ such that $f(P) = -t$ and $f(Q) = t + a$. Then $g(P) = -f(P) = t$ and $h(Q) = f(P) - a = t$, so g and h are both surjective. Moreover, for any $P, Q \in \ell$,

$$|g(P) - g(Q)| = |-f(P) + f(Q)| = |f(P) - f(Q)| = d(P, Q)$$

and

$$|h(P) - h(Q)| = |(f(P) - a) - (f(Q) - a)| = |f(P) - f(Q)| = d(P, Q).$$

Thus g and h are rulers for ℓ.

Ruler Placement Theorem If A and B are points on a line ℓ of a metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$, then there exists a ruler g on ℓ for which $g(A) = 0$ and $g(B) > 0$.

Proof Let $f : \ell \to \mathbb{R}$ be a ruler for ℓ. Let $a = f(A)$ and define $h : \ell \to \mathbb{R}$ by $h(P) = f(P) - a$. Then $h(A) = 0$. If $h(B) > 0$, let $g = h$; if $h(B) < 0$ define $g : \ell \to \mathbb{R}$ by $g(P) = -f(P)$ for all $P \in \mathcal{P}$. Then g is a ruler for ℓ with $g(A) = 0$ and $g(B) > 0$.

Definition If A and B are points in an incidence geometry $\{\mathcal{P}, \mathcal{L}, d\}$, $\ell = \overleftrightarrow{AB}$, and f is a ruler for ℓ with $f(A) = 0$ and $f(B) > 0$, then we call f a *coordinate system with A as origin and B positive.*

Theorem Suppose $f : \ell \to \mathbb{R}$ and $g : \ell \to \mathbb{R}$ are rulers for a line ℓ in a metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$. Then there exists $a \in \mathbb{R}$ such that either

$$g(P) = f(P) - a$$

or

$$g(P) = -(f(P) - a)$$

for all $P \in \ell$.

Proof Let T be the unique point on ℓ with $g(T) = 0$. Let $a = f(T)$. Then for any $P \in \ell$,

$$|g(P)| = |g(P) - g(T)| = d(P, T) = |f(P) - f(T)| = |f(P) - a|.$$

Hence either

$$g(P) = f(P) - a$$

or

$$g(P) = -(f(P) - a).$$

Now suppose there exists distinct points $P \neq T$ and $Q \neq T$ for which $g(P) = f(P) - a$ and $g(Q) = -(f(Q) - a)$. Then

$$d(P, Q) = |g(P) - g(Q)| = |(f(P) - a) + (f(Q) - a)| = |f(P) + f(Q) - 2a|.$$

Hence

$$|f(P) - f(Q)| = d(P, Q) = |f(P) + f(Q) - 2a|,$$

and so either

$$f(P) - f(Q) = f(P) + f(Q) - 2a,$$

in which case $f(Q) = a$, or

$$f(P) - f(Q) = -f(P) - f(Q) + 2a,$$

in which case $f(P) = a$. Either of these conclusions contradict the assumption that f is injective. Hence we must have that either

$$g(P) = f(P) - a$$

for all $P \in \ell$ or

$$g(P) = -(f(P) - a)$$

for all $P \in \ell$.