Lecture 4: Metric Geometry

4.1 Distance

Definition A *distance function* on a set $\mathcal{S}$ is a function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that for all $P, Q \in \mathcal{S}$, (1) $d(P,Q) \geq 0$, (2) $d(P,Q) = 0$ if and only if $P = Q$, and (3) $d(P,Q) = d(Q,P)$.

Example Define $d_E : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as follows: if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$d_E(P,Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2)}.$$

Then $d_E(P,Q) \geq 0$ for all $P, Q \in \mathbb{R}^2$. If $P = (x_1, y_1)$, $Q = (x_2, y_2)$, and $d_E(P,Q) = 0$, then we must have

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 0,$$

from which it follows that

$$(x_1 - x_2)^2 = 0$$

and

$$(y_1 - y_2)^2 = 0.$$

Hence $x_1 = x_2$ and $y_1 = y_2$, that is, $P = Q$. Also, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$d_E(P,Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2)} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2)} = d_E(Q,P).$$

Thus d_E is a distance function, which we call the *Euclidean distance function.*

Example Define $d_T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as follows: if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2|.$$

Then $d_E(P,Q) \geq 0$ for all $P, Q \in \mathbb{R}^2$. If $P = (x_1, y_1)$, $Q = (x_2, y_2)$, and $d_E(P,Q) = 0$, then we must have

$$|x_1 - x_2| + |y_1 - y_2| = 0,$$

from which it follows that

$$|x_1 - x_2| = 0$$

and

$$|y_1 - y_2| = 0.$$

Hence $x_1 = x_2$ and $y_1 = y_2$, that is, $P = Q$. Also, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| + |y_2 - y_1| = d_T(Q,P).$$

Thus d_T is a distance function, which we call the *taxicab distance function.*

4.2 Rulers

Definition Suppose d is a distance function on a set $\mathcal{P}$ and $\{\mathcal{P}, \mathcal{L}\}$ is an incidence geometry. Given a line $\ell \in \mathcal{L}$, we call a function $f : \ell \to \mathbb{R}$ a *ruler*, or *coordinate system*, for ℓ if (1) f is a bijection and (2) for every pair of points $P, Q \in \ell$,

$$d(P, Q) = |f(P) - f(Q)|.$$

We call $f(P)$ the *coordinate* of P with respect to f.

Definition Suppose d is a distance function on $\mathcal{P}$, $\{\mathcal{P}, \mathcal{L}\}$ is an incidence geometry, and every line $\ell \in \mathcal{L}$ has a ruler. We say $\{\mathcal{P}, \mathcal{L}\}$ satisfies the *Ruler Postulate* and we call $\{\mathcal{P}, \mathcal{L}, d\}$ a *metric geometry.*

Note: Suppose $\{\mathcal{P}, \mathcal{L}\}$ is an incidence geometry, d is a distance function on $\mathcal{P}$, $\ell \in \mathcal{L}$, $f : \ell \to \mathbb{R}$ is surjective, and for every $P, Q \in \mathcal{P}$,

$$d(P, Q) = |f(P) - f(Q)|.$$

Then if $P, Q \in \mathcal{P}$ with $f(P) = f(Q)$,

$$d(P, Q) = |f(P) - f(Q)| = 0,$$

from which it follows that $P = Q$ and f is injective (and hence a bijection).

Example Let $\ell = L_a$ be a vertical line in the Cartesian Plane $\{\mathbb{R}^2, \mathcal{L}_E\}$. If $P \in \ell$, then $P = (a, y)$ for some y. Define $f : \ell \to \mathbb{R}$ by

$$f(P) = f((a, y)) = y.$$

Then for any $P = (a, y_1)$ and $Q = (a, y_2)$ on ℓ,

$$|f(P) - f(Q)| = |y_1 - y_2| = d_E(P, Q).$$

Since f is a bijection, it follows that f is a ruler for ℓ.

Now let $\ell = L_{m,b}$ be a non-vertical line in $\{\mathbb{R}^2, \mathcal{L}_E\}$. Note that if

$$P = (x_1, y_1) = (x_1, mx_2 + b)$$

and

$$Q = (x_2, y_2) = (x_2, mx_2 + b)$$

are points on ℓ, then

$$d_E(P, Q) = \sqrt{(x_1 - x_2)^2 + m^2(x_1 - x_2)^2} = \sqrt{1 + m^2}|x_1 - x_2|.$$

Hence it would be reasonable to define $f : \ell \to \mathbb{R}$ at $P = (x, y)$ by

$$f(P) = f((x, y)) = x\sqrt{1 + m^2}.$$

Then for every $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on ℓ,

$$|f(P) - f(Q)| = |x_1\sqrt{1 + m^2} - x_2\sqrt{1 + m^2}| = \sqrt{1 + m^2}|x_1 - x_2| = d(P, Q).$$

Moreover, f is a surjection since, for any $t \in \mathbb{R}$, $f(P) = t$ where

$$P = \left(\frac{t}{\sqrt{1 + m^2}}, \frac{mt}{\sqrt{1 + m^2}} + b \right).$$

Hence f is a ruler for ℓ, and $\mathcal{E} = \{\mathbb{R}^2, \mathcal{L}_E, d_E\}$ is a metric geometry, which we call the *Euclidean Plane*.

Example $\mathcal{T} = \{\mathbb{R}^2, \mathcal{L}_E, d_T\}$ is also a metric geometry, which we call the *Taxicab Plane*. The verification of this is a homework exercise.

4.3 The hyperbolic plane

We will need the following hyperbolic trigonometric functions:

$$\sinh(t) = \frac{e^t - e^{-t}}{2},$$

$$\cosh(t) = \frac{e^t + e^{-t}}{2},$$

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}},$$

and

$$\operatorname{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}}.$$

Note that

$$\frac{d}{dt}\sinh(t) = \cosh(t),$$

$$\frac{d}{dt}\cosh(t) = \sinh(t),$$

$$\cosh^2(t) - \sinh^2(t) = \frac{1}{4}(e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}) = 1,$$

$$\tanh^2(t) + \operatorname{sech}^2(t) = 1,$$

$$\frac{d}{dt}\tanh(t) = \frac{\cosh^2(t) - \sinh^2(t)}{\cosh^2(t)} = \text{sech}^2(t),$$

and

$$\frac{d}{dt}\text{sech}(t) = -(\cosh(t))^{-2}\sinh(t) = -\text{sech}(t)\tanh(t).$$

Moreover,

$$\lim_{t\to-\infty}\tanh(t) = -1$$

and

$$\lim_{t\to\infty}\tanh(t) = 1,$$

combined with the fact that $\frac{d}{dt}\tanh(t) > 0$ for all t, shows that $f(t) = \tanh(t)$ is a bijection from $(-\infty, \infty)$ to $(-1, 1)$. It is also easy to show that $\cosh(t) \geq 1$ for all t, from which it follows that $0 < \text{sech}(t) \leq 1$ for all t.

From the above, the function $\varphi(t) = (\tanh(t), \text{sech}(t))$ parametrizes the upper half of the circle $x^2 + y^2 = 1$, with

$$\varphi(0) = (0, 1),$$

$$\lim_{t\to-\infty}\varphi(t) = (-1, 0),$$

and

$$\lim_{t\to\infty}\varphi(t) = (1, 0).$$

More generally, $\psi(t) = (r\tanh(t) + c, r\text{sech}(t))$ parametrizes the upper half of the circle $(x - c)^2 + y^2 = r^2$, with

$$\psi(0) = (c, r),$$

$$\lim_{t\to-\infty}\psi(t) = (c - r, 0),$$

and

$$\lim_{t\to\infty}\psi(t) = (c + r, 0).$$

Moreover, $\psi : \mathbb{R} \to {}_cL_r$ is a bijection.

We now know that if $P = (x, y)$ lies on ${}_cL_r$, then there exists a unique value of t such $(x, y) = (r\tanh(t) + c, r\text{sech}(t))$. In particular,

$$x = r\tanh(t) + c,$$

so

$$\frac{x - c}{r} = \tanh(t).$$

Now $s = \tanh(t)$ implies

$$s = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{e^{2t} - 1}{e^{2t} + 1} = \frac{u^2 - 1}{u^2 + 1},$$

where $u = e^t$. Hence

$$(1-s)u^2 = 1+s,$$

so

$$u = \sqrt{\frac{1+s}{1-s}},$$

and

$$t = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right).$$

Hence

$$\begin{aligned} t &= \frac{1}{2}\log\left(\frac{1+\frac{x-c}{r}}{1-\frac{x-c}{r}}\right) \\ &= \frac{1}{2}\log\left(\frac{r+(x-c)}{r-(x-c)}\right) \\ &= \frac{1}{2}\log\left(\frac{(r+(x-c))(r+(x-c))}{(r-(x-c))(r+(x-c)})\right) \\ &= \frac{1}{2}\log\left(\frac{(r+(x-c))^2}{r^2-(x-c)^2}\right) \\ &= \frac{1}{2}\log\left(\frac{(r+(x-c)^2}{y^2}\right) \\ &= \log\left(\frac{x-c+r}{y}\right). \end{aligned}$$

We might then think of defining a distance function d_H on the Poincaré Plane $\{\mathbb{H}, \mathcal{L}_H\}$ in such a way that the function $f : {}_cL_r \to \mathbb{R}$ defined for $P = (x, y)$ on ${}_cL_r$ by

$$f(P) = \log\left(\frac{x-c+r}{y}\right)$$

is a ruler. To do so, we simply define d_H for points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on ${}_cL_r$ by

$$\begin{aligned} d_H(P,Q) = |f(P) - f(Q)| &= \left|\log\left(\frac{x_1-c+r}{y_1}\right) - \log\left(\frac{x_2-c+r}{y_2}\right)\right| \\ &= \left|\log\left(\frac{\frac{x_1-c+r}{y_1}}{\frac{x_2-c+r}{y_2}}\right)\right|. \end{aligned}$$

However, it remains to define rulers and distances for points on vertical lines. An analogous bijection from $f : {}_aL \to \mathbb{R}$ is given by, for $P = (a, y)$ on ${}_aL$,

$$f(P) = \log(y),$$

from which we obtain, for $P = (a, y_1)$ and $Q = (a, y_2)$ on ${}_aL$,

$$d_H(P, Q) = \left| \log \left(\frac{y_1}{y_2} \right) \right|.$$

If we include the case $y_1 = y_2$ in this formula, we then have $d_H(P, Q) = 0$ if and only if $P = Q$, and d_H is indeed a distance function on $\mathbb{H}$.

From now on, when we refer to the Poincaré Plane we mean the metric geometry $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H, d_H\}$.

Note that the previous example is an example of the following situation: Suppose $\{\mathcal{P}, \mathcal{L}\}$ is an incidence geometry and assume that for each $\ell \in \mathcal{L}$ there exists a bijection $f_\ell : \ell \to \mathbb{R}$. Then we may define a distance function on $\mathcal{P}$ as follows: Given points $P, Q \in \mathcal{P}$, let $d(P, Q) = 0$ if $P = Q$ and

$$d(P, Q) = |f_\ell(P) - f_\ell(Q)|,$$

where $\ell = \overleftrightarrow{PQ}$, otherwise. It is not hard to show that $\{\mathcal{P}, \mathcal{L}, d\}$ is a metric geometry. This illustrates that metric geometries can be very strange since the choice of bijections can be arbitrary and need not be consistent between different lines.