Lecture 3: Incidence Geometry

3.1 Abstract geometry

Definition Suppose $\mathcal{P}$ is a set and $\mathcal{L}$ is a set of non-empty subsets of $\mathcal{P}$ such that (1) for every $A \in \mathcal{P}$ and $B \in \mathcal{P}$ there exists $\ell \in \mathcal{L}$ such that $A \in \ell$ and $B \in \ell$, and (2) for every $\ell \in \mathcal{L}$, ℓ has at least two elements from $\mathcal{P}$. We call $\{\mathcal{P}, \mathcal{L}\}$ an *abstract geometry*, the elements of $\mathcal{P}$ *points*, and the elements of $\mathcal{L}$ *lines*. Moreover, if $P \in \mathcal{P}$, $\ell \in \mathcal{L}$, and $P \in \ell$, we say P *lies* on ℓ, or that ℓ passes through P.

Note: The conditions in the definition say that every pair of points lies on a line and that every line passes through at least two points.

Example Recall that the unit sphere in $\mathbb{R}^3$ is the set

$$S^2 = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}, x^2 + y^2 + z^2 = 1\}$$

and, for any given real numbers a, b, c, and d, where not all of a, b, and c are 0, the set

$$\{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}, ax + by + cz = d\}$$

is a plane in $\mathbb{R}^3$. If $d = 0$, the plane passes through the origin. Given a plane A passing through the origin, the set $S^2 \cap A$ is a great circle of S^2. Given $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ on S^2, we will show that P and Q both lie on some great circle. To do so, we need to find real numbers a, b, and c, not all 0, such that

$$ax_1 + by_1 + cz_1 = 0$$

and

$$ax_2 + by_2 + cz_2 = 0.$$

Since this is a homogeneous system in three unknowns, we know it is possible to solve for a, b, and c, not all 0. Thus if we let $\mathcal{L}_R$ be the set of great circles of S^2, then $\{S^2, \mathcal{L}_R\}$ is an abstract geometry, called the *Riemann sphere*.

Note that on the Riemann sphere, the points $P = (0, 0, 1)$ (the "north pole") and $Q = (0, 0, -1)$ (the "south pole") lie both on the great circle determined by the plane with equation $x = 0$ and the the great circle determined by the plane $y = 0$. That is, in this geometry, two distinct points do not determine a unique line.

3.2 Incidence geometry

Definition Given an abstract geometry $\{\mathcal{P}, \mathcal{L}\}$, a set of points $S \subset \mathcal{P}$ is *collinear* if there exists a line $\ell \in \mathcal{L}$ such that $S \subset \mathcal{L}$. A set of points which is not collinear is *non-collinear*.

Definition An abstract geometry $\{\mathcal{P}, \mathcal{L}\}$ is an *incidence geometry* if (1) distinct points P and Q in $\mathcal{P}$ lie on a unique line, and (2) there exists a set of three non-collinear points.

Example Given a real number a, let

$$L_a = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, x = a\}.$$

Given real numbers m and b, let

$$L_{m,b} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, y = mx + b\}.$$

Let

$$\mathcal{L}_E = \{L_a : a \in \mathbb{R}\} \cup \{L_{m,b} : m \in \mathbb{R}, b \in \mathbb{R}\}.$$

We will show that $\{\mathbb{R}^2, \mathcal{L}_E\}$ is an incidence geometry. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points in $\mathbb{R}^2$. If $x_1 = x_2$, let $a = x_1$. Then $P \in L_a$ and $Q \in L_a$. Clearly, there is no $b \neq a$ for which $P \in L_b$. Suppose there exist real numbers m and b for which $P \in L_{m,b}$ and $Q \in L_{m,b)}$. Then

$$y_1 = mx_1 + b = ax + b$$

and

$$y_2 = mx_2 + b = ax + b.$$

But then

$$P = (x_1, y_1) = (a, ax + b) = (a, y_2) = (x_2, y_2) = Q,$$

contradicting our assumption that P and Q are distinct points.

Now suppose $x_1 \neq x_2$. We need to find m and b such that P and Q both lie on $L_{m,b}$. That is, we want

$$y_1 = mx_1 + b$$

and

$$y_2 = mx_2 + b.$$

Subtracting, we have

$$y_2 - y_1 = m(x_2 - x_1),$$

from which we have, since $x_2 - x_1 \neq 0$,

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We then have

$$b = y_1 - mx_1 = y_2 - mx_2.$$

Note that since the solutions are unique, P and Q lie on a unique line $L_{m,b}$.

You will show in the homework that there exists a set of three non-collinear points. Hence $\mathcal{E} = \{\mathbb{R}^2, \mathcal{L}_E\}$ is an incidence geometry, which we call the *Cartesian Plane.*

We call the lines L_a *vertical lines* and the lines $L_{m,b}$ *non-vertical lines.*

Example Let $\mathbb{H} = \{(x, y) : (x, y) \in \mathbb{R}^2, y > 0\}$. Given any real number a, let ${}_aL = \{(x, y) : (x, y) \in \mathbb{H}, x = a\}$, and given any real numbers c and $r > 0$ let

$$ {}_cL_r = \{(x, y) : (x, y) \in \mathbb{H}, (x - c)^2 + y^2 = r^2\}. $$

Let

$$ \mathcal{L}_H = \{a_L : a \in \mathbb{R}\} \cup \{{}_cL_r : c \in \mathbb{R}, r > 0\}. $$

We will show that $\{\mathbb{H}, \mathcal{L}_H\}$ is an incidence geometry. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points in $\mathbb{H}$. If $x_1 = x_2$, let $a = x_1$. Then P and Q both lie on ${}_aL$. Suppose P and Q also lie on ${}_cL_r$. Then we must have

$$ (x_1 - c)^2 + y_1^2 = r^2 $$

and

$$ (x_2 - c)^2 + y_2^2 = r^2. $$

Hence, since $a = x_1 = x_2$,

$$ y_1^2 = r^2 - (x_1 - c)^2 = r^2 - (a - c)^2 = r^2 - (x_2 - c)^2 = y_2^2. $$

Since $y_1 > 0$ and $y_2 > 0$, it follows that $y_1 = y_2$, and $P = Q$, contradicting our assumption that P and Q are distinct points in $\mathbb{H}$.

Now suppose $x_1 \neq x_2$. We need to find c and $r > 0$ such that

$$ (x_1 - c)^2 + y_1^2 = r^2 $$

and

$$ (x_2 - c)^2 + y_2^2 = r^2. $$

Subtracting and expanding gives us

$$ x_1^2 - 2x_1c - x_2^2 + 2x_2c = y_2^2 - y_1^2, $$

from which it follows that

$$ c = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)}. $$

We then have

$$ r = \sqrt{(x_1 - c)^2 + y_1^2} = \sqrt{(x_2 - c)^2 + y_2^2}. $$

Thus P and Q lie on a unique line ${}_cL_r$.

You will show in the homework that there exists a set of three non-collinear points. Hence $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ is an incidence geometry, which we call the *Poincaré Plane*.

We call the lines ${}_aL$ *type I lines* and the lines ${}_cL_r$ *type II lines*.

Notation: Given distinct points P and Q in an incidence geometry, we let $\overleftrightarrow{PQ}$ denote the unique line on which both P and Q lie.

Theorem If ℓ_1 and ℓ_2 are lines in an incidence geometry and $\ell_1 \cap \ell_2$ has two or more distinct points, then $\ell_1 = \ell_2$.

Proof If P and Q are distinct points in $\ell_1 \cap \ell_2$, then

$$\ell_1 = \overleftrightarrow{PQ} = \ell_2.$$

Definition We say lines ℓ_1 and ℓ_2 in an abstract geometry are *parallel*, denoted $\ell_1 \| \ell_2$, if either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$.

Theorem In an incidence geometry, two distinct lines are either parallel or they intersect in exactly one point.