Lecture 25: The Critical Function

25.1 The critical number

Theorem If, in a neutral geometry, ℓ is a line, P is a point, $P \notin \ell$, D is the foot of the perpendicular from P to ℓ, and C is a point with $m(\angle DPC) \geq 90$, then $\overrightarrow{PC} \cap \ell = \emptyset$.

Proof If $m(\angle DPC) = 90$, then $\overleftrightarrow{PC}$ is parallel to ℓ. If $m(\angle DBC) > 90$, then let A be a point on the same side of $\overleftrightarrow{PD}$ as C with $m(\angle DPA) = 90$. Then $A \in \text{int}(\angle PDC)$, so D and C lie on opposite sides of $\overleftrightarrow{PA}$. Since $\overleftrightarrow{PA}$ is parallel to ℓ, it follows that $\text{int}(\overrightarrow{PC})$ and ℓ are on opposite sides of $\overleftrightarrow{PA}$. Hence $\overrightarrow{PC} \cap \ell = \emptyset$.

Definition Given a nonempty set S of real numbers, we call r the *supremum*, or *least upper bound*, of S if (1) $s \leq r$ for all $s \in S$, and (2) if $t < r$, then there exists $s \in S$ such that $t < r$. The supremum of a set S is denoted either $\sup S$ or lub S.

Example If $S = \{x : x \in \mathbb{Q}, x^2 \leq 2\}$, then $\sup S = \sqrt{2}$.

Example If $S = \{x : x \in \mathbb{R}, 0 < x \leq 3\}$, then $\sup S = 3$.

Note: If S is a nonempty, bounded set of real numbers, then S has a unique least upper bound.

Definition In a neutral geometry, given a line ℓ, a point P, $P \notin \ell$, and D the foot of the perpendicular from P to ℓ, let

$$K(P, \ell) = \{r : r \in \mathbb{R}, r = m(\angle DPC) \text{ where } \overrightarrow{PC} \cap \ell \neq \emptyset\}.$$

We call

$$r(P, \ell) = \sup K(P, l)$$

the *critical number* for P and ℓ.

Example In the Euclidean Plane, $r(P, \ell) = 90$ for all lines ℓ and points $P \notin \ell$.

Theorem If, in a neutral geometry, P is a point not on line ℓ, D is the foot of the perpendicular from P to ℓ, and $m(\angle DPC) \geq r(P, \ell)$, then $\overrightarrow{PC} \cap \ell = \emptyset$. If $m(\angle DPC) < r(P, \ell)$, then $\overrightarrow{PC} \cap \ell \neq \emptyset$.

Proof Suppose $m(\angle DPC) = r(P, \ell)$ and $\overrightarrow{PC} \cap \ell \neq \emptyset$. Let $\overrightarrow{PC} \cap \ell = \{R\}$ and let S be a point with $D - R - S$. Then $R \in \text{int}(\angle DPS)$, so $m(\angle DPS) > r(P, \ell)$, contradicting the fact that $m(\angle DPS) \in K(P, \ell)$ and $r(P, \ell) = \sup K(P, \ell)$. Note that if $B - P - C$, then $\overrightarrow{PB} \cap \ell = \emptyset$ since $m(\angle DPB) \geq 90$. Hence, in fact, if $m(\angle DPC) = r(P, \ell)$, then $\overleftrightarrow{PC}$ is parallel to ℓ.

Now suppose $m(\angle DPC) > r(P, \ell)$. Let E be a point on the same side of $\overleftrightarrow{PD}$ as C for which $m(\angle DPE) = r(P, \ell)$. Then $\overleftrightarrow{PE}$ is a parallel to ℓ. Now $E \in \text{int}(\angle DPC)$, so D and C are on opposite sides of $\overleftrightarrow{PE}$. Hence $\overrightarrow{PC}$ and ℓ are on opposite sides of $\overleftrightarrow{PE}$, and so $\overrightarrow{PC} \cap \ell = \emptyset$.

Now suppose $m(\angle DPC) < r(P, \ell)$. Then there exists an $s \in \mathbb{R}$ and a point F such that $s = m(\angle DPF)$ and $\overrightarrow{PF} \cap \ell \neq \emptyset$. Let $\{A\} = \overrightarrow{PF} \cap \ell$. If A is on the same side of $\overleftrightarrow{PD}$ as C, then $C \in \text{int}(\angle DPF)$. Hence, by Crossbar, $\overrightarrow{PC} \cap \overline{DA} \neq \emptyset$, and so $\overrightarrow{PC} \cap \ell \neq \emptyset$. If A and C are on opposite sides of $\overleftrightarrow{PD}$, let A' be the point on ℓ with $A - D - A'$ and $\overline{AD} \simeq DA'$. Then $\triangle ADP \simeq \triangle A'DP$ by Side-Angle-Side. In particular, $\angle DPA' \simeq \angle DPA$. Hence $C \in \text{int}(\angle DPA')$, and so, as above, $\overrightarrow{PC} \cap \overline{A'D} \neq \emptyset$. Hence $\overrightarrow{PC} \cap \ell \neq \emptyset$.

Note: We now have, for $C \notin \overleftrightarrow{PD}$, $\overrightarrow{PC} \cap \ell$ if and only if $m(\angle DPC) < r(P, \ell)$.

Theorem If, in a neutral geometry, ℓ is a line and P is a point not on ℓ, then there exist two or more lines through P parallel to ℓ if and only if $r(P, \ell) < 90$.

Proof See homework.

25.2 The critical function

Theorem If, in a neutral geometry, ℓ and m are lines, P and Q are points, $P \notin \ell$, $Q \notin m$, and $d(P, \ell) = d(Q, m)$, then $r(P, \ell) = r(Q, \ell)$.

Proof The result will follow if we prove that $K(P, \ell) = K(Q, m)$. Let D be the foot of the perpendicular from P to ℓ and F be the foot of the perpendicular from Q to m. Then $\overline{PD} \simeq \overline{QF}$. If $s \in K(P, \ell)$, then there exists a point $C \in \ell$ with $m(\angle DPC) = s$. Let G be a point on m with $\overline{DC} \simeq \overline{FG}$. Then $\triangle PDC \simeq \triangle QFG$ by Side-Angle-Side. In particular, $m(\angle FQG) = m(\angle DPC)$, so $s \in K(Q, m)$. Hence $K(P, \ell) \subset K(Q, m)$. A similar argument shows that $K(Q, m) \subset K(P, \ell)$, and so $K(Q, m) = K(P, \ell)$. Thus $r(P, \ell) = r(Q, m)$.

Definition In a neutral geometry, we call the function $\Pi : (0, \infty) \to (0, 90]$ given by

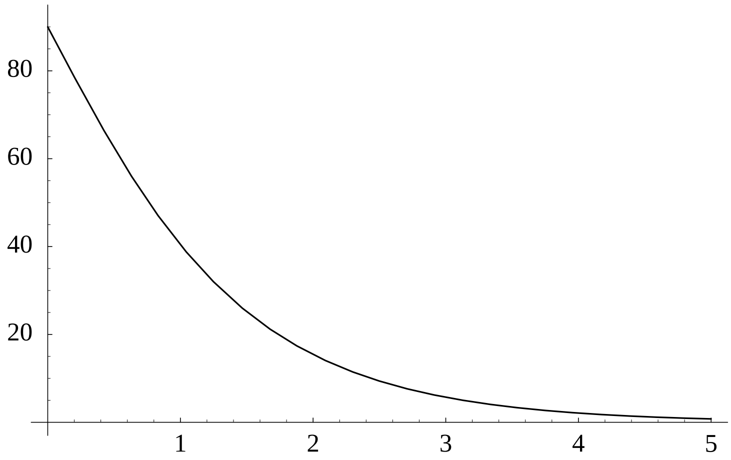
$$\Pi(t) = r(P, \ell),$$

where ℓ is a line and P is a point with $d(P, \ell) = t$, the *critical function.*

Definition For the Euclidean Plane, $\Pi(t) = 90$ for all $0 < t < \infty$.

Example For the Poincaré Plane, one may show that

$$\Pi(t) = \tan^{-1}\left(\frac{1}{\sinh(t)}\right).$$



$\Pi(t)$ for the Poincaré Plane

Theorem In a neutral geometry, $\Pi(s) \leq \Pi(t)$ whenever $s > t$.

Proof Let ℓ be a line, $D \in \ell$, and P and Q two points such that $Q - P - D$, $\overleftrightarrow{QD} \perp \ell$, $QD = s$, and $PD = t$. Let C and E be points on the same side of $\overleftrightarrow{PD}$ with $m(\angle DPC) = \Pi(t)$ and $m(\angle DQE) = \Pi(t)$. Then $\overleftrightarrow{PC}$ is parallel to ℓ and $\overleftrightarrow{PC}$ is parallel to $\overleftrightarrow{QE}$ (since $\overleftrightarrow{QD}$ is a transversal of $\overleftrightarrow{PC}$ and $\overleftrightarrow{QE}$ with a pair of congruent corresponding angles, and hence congruent alternating interior angles). Since Q and D are on opposite sides of $\overleftrightarrow{PC}$, it follows that $\overleftrightarrow{QE} \cap \ell = \emptyset$. In particular, $\overrightarrow{QE} \cap \ell = \emptyset$, so

$$\Pi(s) \leq m(\angle DQE) = \Pi(t).$$

Theorem If, in a neutral geometry, $\Pi(a) < 90$ for some $a \in (0, \infty)$, then $\Pi\left(\frac{a}{2}\right) < 90$.

Proof Let ℓ be a line, $D \in \ell$, P a point with $\overleftrightarrow{PD} \perp \ell$ and $PD = a$, and Q the midpoint of $\overline{PD}$. Let C be a point with $m(\angle DPC) = \Pi(a)$ and let m be the unique line through Q perpendicular to $\overleftrightarrow{PD}$.

If $\overrightarrow{PC} \cap m = \emptyset$, then

$$\Pi\left(\frac{a}{2}\right) = r(P, m) \leq m(\angle QPC) = \Pi(a) < 90.$$

So suppose $\overrightarrow{PC} \cap m = \{A\}$. Let B be a point with $P - A - B$. Then $B \in \text{int}(\angle DQA)$, so

$$m(\angle DQB) < m(\angle DQA) = 90.$$

Since $\overrightarrow{PA} \cap \ell = \emptyset$, P and B are on the same side of ℓ. Since $P - Q - D$, P and Q are on the same side of ℓ. Hence B and Q are on the same side of ℓ, so $\overline{QB} \cap \ell = \emptyset$. If E is a point with $Q - B - E$, then Q and E are on opposite sides of $\overleftrightarrow{PC}$. Since $P - Q - D$, Q and D are on the same side of $\overleftrightarrow{PC}$. Hence E and D are on opposite sides of $\overleftrightarrow{PC}$. Thus $\overrightarrow{BE} \cap \ell = \emptyset$, and so $\overrightarrow{QB} \cap \ell = \emptyset$. Thus

$$\Pi\left(\frac{a}{2}\right) = r(Q, \ell) \leq m(\angle DQB) < 90.$$

Theorem If, in a neutral geometry, $\Pi(a) < 90$ for some $a \in (0, \infty)$, the $\Pi(t) < 90$ for all $t > 0$.

Proof Let $t \in (0, \infty)$. If $t \geq a$, then

$$\Pi(t) \leq \Pi(a) < 90.$$

If $t < a$, let n be an integer for which

$$\frac{a}{2^n} < t.$$

Then

$$\Pi(t) \leq \Pi\left(\frac{a}{2^n}\right) < 90.$$

All or None Theorem If, in a neutral geometry, there exists a line ℓ and a point P not on ℓ for which there is a unique line through P parallel to ℓ, then the Euclidean Parallel Property holds.

Definition We say a neutral geometry satisfies the *Hyperbolic Parallel Property* if for each line ℓ and point $P \notin \ell$ there exist two or more lines through P parallel to ℓ.

Definition We call a neutral geometry satisfying the Euclidean Parallel Property a *Euclidean geometry.* We call a neutral geometry satisfying the Hyperbolic Parallel Property a *hyperbolic geometry.*

Note: A given neutral geometry must be either a Euclidean geometry or a hyperbolic geometry.