Lecture 24: Saccheri Quadrilaterals

24.1 Saccheri Quadrilaterals

Definition In a protractor geometry, we call a quadrilateral $\square ABCD$ a *Saccheri quadrilateral*, denoted $\boxed{\text{S}}ABCD$, if $\angle A$ and $\angle D$ are right angles and $\overline{AB} \simeq \overline{CD}$. Given $\boxed{\text{S}}ABCD$, we call $\overline{AD}$ the *lower base*, $\overline{BC}$ the *upper base*, $\overline{AB}$ and $\overline{CD}$ the *legs*, $\angle A$ and $\angle D$ the *lower base angles*, and $\angle B$ and $\angle C$ the *upper base angles*.

Theorem A Saccheri quadrilateral is a convex quadrilateral.

Proof Given $\boxed{\text{S}}ABCD$, $\overleftrightarrow{AD}$ is perpendicular to both $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Hence $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, and so $\boxed{\text{S}}ABCD$ is a convex quadrilateral.

Definition Given convex quadrilaterals $\square ABCD$ and $\square PQRS$, we write $\square ABCD \simeq \square PQRS$ if $\overline{AB} \simeq \overline{PQ}$, $\overline{BC} \simeq \overline{QR}$, $\overline{CD} \simeq \overline{RA}$, $\overline{DA} \simeq \overline{SP}$, $\angle A \simeq \angle P$, $\angle B \simeq \angle Q$, $\angle C \simeq \angle R$, and $\angle D \simeq \angle S$. We say two convex quadrilaterals are *congruent* it it is possible to label the vertices of one with A, B, C, and D and the other with P, Q, R, and S in such a way that $\square ABCD \simeq \square PQRS$.

Theorem If, in a neutral geometry, $\overline{AD} \simeq \overline{PS}$ and $\overline{AB} \simeq \overline{PQ}$, then $\boxed{\text{S}}ABCD \simeq \boxed{\text{S}}PQRS$.

Proof See homework.

Theorem Given $\boxed{\text{S}}ABCD$ in a neutral geometry, $\boxed{\text{S}}ABCD \simeq DCBA$. In particular, $\angle B \simeq \angle C$.

Proof See homework.

Polygon Inequality If P_1, P_2, $\ldots$, P_n are points in a neutral geometry, then

$$P_1P_n \leq P_1P_2 + P_2P_3 + \cdots P_{n-1}P_n.$$

Proof The result is the Triangle Inequality for $n = 3$. For $n = 4$,

$$P_1P_3 \leq P_1P_2 + P_2P_3,$$

and

$$P_1P_4 \leq P_1P_3 + P_3P_4,$$

so

$$P_1P_4 \leq P_1P_2 + P_2P_3 + P_3P_4.$$

The result follows from repeating the process.

Theorem Given $\square ABCD$ in a neutral geometry, $\overline{BC} \geq \overline{AD}$.

Proof Let $A_1 = A$, $B_1 = B$, $A_2 = D$, and $B_2 = C_2$. For each $k > 2$, let A_k be the unique point on $\overleftrightarrow{AD}$ such that $A_{k-2} - A_{k-1} - A_k$ and $\overline{A_{k-1}A_k} \simeq \overline{AD}$, and let B_k be the unique point on the same side of $\overleftrightarrow{AD}$ as B with $\overleftrightarrow{A_kB_k} \perp \overleftrightarrow{AD}$ and $\overline{A_kB_k} \simeq \overline{BA}$. Then $\square ABCD \simeq \square A_kB_kA_{k+1}B_{k+1}$ for $k = 1, 2, 3, \ldots$.

Hence, for any $n \geq 1$,

$$A_1A_{n+1} \leq A_1B_1 + B_1B_2 + B_2B_3 + \cdots + B_nB_{n+1} + B_{n+1}A_{n+1},$$

and so

$$nAD \leq 2AB + nBC$$

for every $n \geq 1$. That is,

$$AD - BC \leq \frac{2}{n}AB$$

for every $n \geq 2$. Thus we must have $AD - BC \leq 0$, so $\overline{AD} \leq \overline{BC}$.

Theorem Given $\square ABCD$ in a neutral geometry, $\angle ABD < \angle BDC$.

Proof See homework.

Theorem In a neutral geometry, the sum of the measures of the acute angles of a right triangle is less than or equal to 90.

Proof Let $\triangle ABD$ be a right triangle with right angle at A. Let C be the unique point on the same side of $\overleftrightarrow{AD}$ as B with $\overleftrightarrow{CD} \perp \overleftrightarrow{AD}$ and $\overline{AB} \simeq \overline{CD}$. Then $\square ABCD$, and so

$$m(\angle ABD) + m(\angle ADB) \leq m(\angle BDC) + m(\angle ADB).$$

Now $B \in \text{int}(\angle ADC)$ (since $\square ABCD$ is a convex quadrilateral), so

$$m(\angle BDC) + m(\angle ADB) = m(\angle ADC) = 90.$$

Hence

$$m(\angle ABD) + m(\angle ADB) \leq 90.$$

Saccheri's Theorem Given $\triangle ABC$ in a neutral geometry,

$$m(\angle A) + m(\angle B) + m(\angle C) \leq 180.$$

Proof Suppose $\overline{AC}$ is a longest side $\triangle ABC$ and let D be the foot of the perpendicular from B to $\overleftrightarrow{AC}$. Then $A - D - C$ and $D \in \text{int}(\angle ABC)$. Hence

$$\begin{aligned} m(\angle CAB) + m(\angle ABC) + m(\angle BCA) &= m(\angle DAB) + m(\angle ABD) \\ &\quad + m(\angle DBC) + m(\angle BCD) \\ &\leq 90 + 90 = 180. \end{aligned}$$

Theorem Given $\triangle ABC$ in a neutral geometry which also satisfies the Euclidean Parallel Property,

$$m(\angle A) + m(\angle B) + m(\angle C) = 180.$$

Proof Let ℓ be the unique line through B parallel to $\overleftrightarrow{AC}$ and let D and E be points with $D - B - E$ and A and D on the same side of $\overleftrightarrow{BC}$. Then $\angle DBA$ and $\angle BAC$ are alternate interior angles, and hence congruent, as are $\angle EBC$ and $\angle ACB$.

Now $A \in \text{int}(\angle DBC)$, so

$$m(\angle DBA) + m(\angle ABC) = m(\angle DBC).$$

Hence

$$\begin{aligned} m(\angle CAB) + m(\angle ABC) + m(\angle BCA) &= m(\angle DBA) + m(\angle ABC) + m(\angle EBC) \\ &= m(\angle DBC) + m(\angle EBC) = 180. \end{aligned}$$

Definition We call $\square ABCD$ a *parallelogram* if $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \| \overleftrightarrow{BC}$. We call $\square ABCD$ a *rectangle* if all four angles are right angles, and we call $\square ABCD$ a *square* if it is a rectangle and

$$\overline{AB} \simeq \overline{BC} \simeq \overline{CD} \simeq \overline{DA}.$$

Theorem Given $\boxed{S}ABCD$ in a neutral geometry with M the midpoint of $\overline{AD}$ and N the midpoint of $\overline{BC}$, then $\overleftrightarrow{MN} \perp \overleftrightarrow{AD}$ and $\overleftrightarrow{MN} \perp \overleftrightarrow{BC}$.

Proof See homework

Theorem In a neutral geometry, a Saccheri quadrilateral is a parallelogram.

Theorem If, in a neutral geometry, $\square ABCD$ has right angles at A and D and $\overline{AB} > \overline{DC}$, then $\angle ABC < \angle DCB$.

Proof Let E be point on $\overrightarrow{DC}$ so that $D - C - E$ and $\overline{DE} \simeq \overline{AB}$. Then $\boxed{\text{S}}ABED$, so $\angle ABE \simeq \angle DEB$. Now $\angle DCB$ is an exterior angle of $\triangle BEC$, so $\angle DCB > \angle DEB$. Moreover, $C \in \text{int}(\angle ABE)$, so $\angle ABC < \angle ABE$. Thus $\angle ABC < \angle DCB$.

Theorem If, in a neutral geometry, $\square ABCD$ has right angles at A and D, then (1) $\overline{AB} > \overline{CD}$ if and only if $\angle ABC < \angle DCB$, (2) $\overline{AB} \simeq \overline{CD}$ if and only if $\angle ABC \simeq \angle DCB$, and (3) $\overline{AB} < \overline{CD}$ if and only if $\angle ABC > \angle DCB$.

Proof See homework.

Definition We say a set of points $\mathcal{S}$ in a neutral geometry $\{\mathcal{P}, \mathcal{L}, d, m\}$ is *equidistant* from a line ℓ if $d(P, \ell) = d(Q, \ell)$ for all $P, Q \in \mathcal{S}$.

Giordano's Theorem If, in a neutral geometry, ℓ and m are lines and A, B, and C are points on ℓ such that

$$d(A, m) = d(B, m) = d(C, m),$$

then ℓ is equidistant from m.

Proof We may assume $\ell \neq m$. At least two of A, B, and C lie on one side of m; suppose A and B lie on the same side of m. If D is the foot of the perpendicular from A to m and E is the foot of the perpendicular from B to m, then $\boxed{\text{S}}DABE$, so $\overleftrightarrow{AB} = \ell$ is parallel to m. Hence C lies on the same side of m as A and B. Let F be the foot of the perpendicular from C to m.

Suppose $A - B - C$. Then we have $\boxed{\text{S}}DABE$, $\boxed{\text{S}}DACF$, and $\boxed{\text{S}}EBCF$. Hence

$$\angle ABE \simeq \angle BAD \simeq BCF \simeq \angle CBE.$$

Since $\angle ABE$ and $\angle CBE$ form a linear pair, it follows that $\angle ABE$ is a right angle, and hence all three Saccheri quadrilaterals are in fact rectangles.

Given $P \in \ell$, other than A, B, or C, we need to show $d(P, m) = AD$. Let S be the foot of the perpendicular from P to m. First suppose P lies between A and C. If $\overleftrightarrow{PS} \perp \ell$, then $\angle APS \simeq \angle PAD$, and so $PS = AD$. If $\overleftrightarrow{PS}$ is not perpendicular to ℓ, then one of $\angle APS$ and $\angle CPS$ is acute. Suppose $\angle APS$ is acute and $\angle CPS$ is obtuse. Then $AD < PS < CF$, contradicting the assumption that $AD = CF$. Hence $\overleftrightarrow{PS} \perp \ell$, and $d(P, m) = d(A, m)$.

Finally, suppose $P \notin \overline{AC}$. Let Q be the point on ℓ such that $P - A - Q$ and $\overline{AQ} \simeq \overline{PA}$ and let T be the foot of the perpendicular from Q to m. Now $\triangle PAD \simeq \triangle QAD$ by Side-Angle-Side and $\triangle PDS \simeq QDT$ by Hypotenuse-Angle. Hence $PS = QT$. Similarly, we may find point R on ℓ such that $P - C - R$ and $\overline{PC} \simeq \overline{CR}$. If U is the foot of the perpendicular from R to m, then, using the same argument, $PS = RU$. Hence

$$d(P, m) = d(Q, m) = d(R, m).$$

Since $P - A - Q$, it follows from above that $PS = AD$. Hence $d(P, m) = d(A, m)$, and ℓ is equidistant from m.