Lecture 23: Parallel Lines

23.1 Transversals

Definition Given distinct lines ℓ, m, and n, we say a line ℓ is a *transversal* of m and n if $\ell \cap m \neq \emptyset$, $\ell \cap n \neq \emptyset$, and $\ell \cap m \neq \ell \cap n$.

Definition Suppose $\overleftrightarrow{GH}$ is a transversal of $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$ in a Pasch geometry with $\overleftrightarrow{GH} \cap \overleftrightarrow{AC} = \{B\}$ and $\overleftrightarrow{GH} \cap \overleftrightarrow{DF} = \{E\}$. Moreover, suppose $A - B - C$, $D - E - F$, $G - B - E - H$, and A and D are on the same side of $\overleftrightarrow{GH}$. Then we call $\angle ABE$ and FEB a pair of *alternate interior angles* and we call $\angle ABG$ and $\angle DEB$ a pair of *corresponding angles.*

Note: In the preceding definition, $\angle DEB$ and $\angle CBE$ are also alternate interior angles, and $\angle CBG$ and $\angle FEB$, $\angle ABE$ and $\angle DEH$, and $\angle CBE$ and $\angle FEH$ are pairs of corresponding angles.

Theorem If, in a neutral geometry, ℓ is a transversal of lines m and n with a pair of alternate interior angles congruent, then there exists a line ℓ' which is perpendicular to both m and n.

Proof Suppose $m = \overleftrightarrow{AC}$, $n = \overleftrightarrow{DF}$, $\ell = \overleftrightarrow{GH}$, $m \cap \ell = \{B\}$, $n \cap \ell = \{E\}$, $A - B - C$, $D - E - F$, $G - B - E - H$, and A and D are on the same side of $\overleftrightarrow{GH}$. If the congruent alternate interior angles are right angles, then we can let $\ell' = \ell$. Otherwise, we may select a pair of congruent alternate interior angles which are acute.

So suppose $\angle ABE \simeq \angle FEB$ and $\angle ABE$ is acute. Let M be the midpoint of $\overline{EB}$, let P be the foot of the perpendicular from M to $\overleftrightarrow{AC}$, and let Q be the foot of the perpendicular from M to $\overleftrightarrow{DF}$. Since $\angle ABM$ and $\angle FEM$ are acute, A and P lie on the same side of $\overleftrightarrow{GH}$ and Q and F lie on the same side of $\overleftrightarrow{GH}$ (see homework). Hence P and Q are on opposite sides of $\overleftrightarrow{GH}$. We need to show that $\overleftrightarrow{PM} = \overleftrightarrow{QM}$.

Now $\triangle MBP \simeq \triangle MEQ$ by Hypotenuse-Angle. In particular, $\angle BMP \simeq \angle EMQ$. Now if $R \in \overleftrightarrow{PM}$ with $P - M - R$, then $\angle BMP \simeq \angle EMR$ (since they are vertical angles). Hence $\angle EMQ \simeq \angle EMR$. Since Q and R are on the same side of $\overleftrightarrow{GH}$, $Q \in \overrightarrow{MR}$. Hence $\overleftrightarrow{PM} = \overleftrightarrow{MQ}$, and we may let $\ell' = \overleftrightarrow{PQ}$.

23.2 Parallel lines

Theorem If, in a neutral geometry, ℓ, m, and n are distinct lines with $n \perp \ell$ and $n \perp m$, then ℓ and m are parallel.

Proof Let $n \cap \ell = \{P\}$, $n \cap m = \{Q\}$. If $\ell \cap m = \{R\}$, then $\triangle PQR$ would have two right angles. Hence $\ell \cap m = \emptyset$, that is, ℓ and m are parallel.

Theorem If, in a neutral geometry, lines ℓ and m have a transversal with congruent alternate interior angles, then ℓ and m are parallel.

Note that in the Poincaré Plane, the lines ${}_0L$ and ${}_1L_1$ are parallel but do not have a common perpendicular

Theorem If, in a neutral geometry, P is a point, ℓ is a line, and $P \notin \ell$, then there exists a line through P parallel to ℓ.

Proof Let Q be the foot of the perpendicular from P to ℓ and let m be the unique line through P perpendicular to $\overleftrightarrow{PQ}$. Then $\overleftrightarrow{PQ} \perp m$ and $\overleftrightarrow{PQ} \perp \ell$, and so m and ℓ are parallel.

23.3 Euclid's parallel postulate

Definition We say a protractor geometry $\{\mathcal{P}, \mathcal{L}, d, m\}$ satisfies *Euclid's Fifth Postulate* (EFP) if whenever $\overleftrightarrow{BC}$ is a transversal of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ with A and D on the same side of $\overleftrightarrow{BC}$ and $m(\angle ABC) + m(\angle(BCD) < 180$, then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point E on the same side of $\overleftrightarrow{BC}$ as A and D.

Theorem If, in a neutral geometry which also satisfies Euclid's Fifth Postulate, P is a point, ℓ is a line, and $P \notin \ell$, then there exists a unique line through P parallel to ℓ.

Proof Let Q be the foot of the perpendicular from P to ℓ and let m be the unique line through P perpendicular to $\overleftrightarrow{PQ}$. Then m is parallel to ℓ, as above. Now suppose $\overleftrightarrow{AB}$ is another line through P. Since $\overleftrightarrow{AB}$ is not perpendicular to $\overleftrightarrow{PQ}$, one of $\angle APQ$ or $\angle BPQ$ is acute. Since $\overleftrightarrow{PQ}$ is perpendicular to ℓ, Euclid's Fifth Postulate implies that $\overleftrightarrow{AB} \cap \ell \neq \emptyset$.

Definition We say an incidence geometry satisfies the *Euclidean Parallel Property*, denoted EPP, or *Playfair's Parallel Postulate*, if for any line ℓ and any point P there exists a unique line through P parallel to ℓ.

We have already seen that if a neutral geometry satisfies Euclid's Fifth Postulate, then it satisfies the Euclidean Parallel Property. The next theorem says these axioms are equivalent.

Theorem A neutral geometry which satisfies the Euclidean Parallel Property also satisfies Euclid's Fifth Postulate.

Proof Let $\overleftrightarrow{BC}$ be a transversal of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ with A and D on the same side of $\overleftrightarrow{BC}$ and $m(\angle ABC) + m(\angle BCD) < 180$. Let E be a point on the same side of $\overleftrightarrow{BC}$ as A such that $\angle EBC$ and $\angle BCD$ are supplementary. Let F be a point with $F - B - E$. Then $\angle FBC$ and $\angle EBC$ are a linear pair, hence supplementary, and so $\angle FBC \simeq \angle BCD$. Hence $\overleftrightarrow{BE}$ and $\overleftrightarrow{CD}$ are parallel. Since $\overrightarrow{BA} \neq \overrightarrow{BE}$, it follows that $\overleftrightarrow{AB}$ is not parallel to $\overleftrightarrow{CD}$. Hence $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} \neq \emptyset$. It remains to show that $\overrightarrow{BA} \cap \overrightarrow{CD} \neq \emptyset$.

Now $\angle CBA < \angle CBE$, so $A \in \mathrm{int}(\angle CBE)$. In particular, A and C are on the same side of $\overleftrightarrow{BE}$. Hence $\mathrm{int}(\overrightarrow{BA})$ and $\overleftrightarrow{CD}$ are on the same side of $\overleftrightarrow{BE}$. Thus $\overrightarrow{BA} \cap \overleftrightarrow{CD} \neq \emptyset$. Finally, since $\mathrm{int}(\overrightarrow{BA})$ and $\mathrm{int}(\overrightarrow{CD})$ lie on the same side of $\overleftrightarrow{BC}$, we must have $\overrightarrow{BA} \cap \overrightarrow{CD} \neq \emptyset$.