Lecture 22: Circles

22.1 Circles

Definition Given a point C in a metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$ and real number $r > 0$, we call

$$\mathcal{C}_r(C) = \{P : P \in \mathcal{P}, PC = r\}$$

a *circle* with *center* C and *radius* r. If $A, B \in \mathcal{C}_r(C)$, we call $\overline{AB}$ a *chord* of $\mathcal{C}_r(C)$; if $C \in \overline{AB}$, and $\overline{AB}$ is a chord, we call $\overline{AB}$ a *diameter* of $\mathcal{C}_r(C)$. If $P \in \mathcal{C}_r(C)$, we call $\overline{PC}$ a *radius segment* of $\mathcal{C}_r(C)$.

Example In the Poincaré Plane, if $C = (a, b)$, then

$$\mathcal{C}_r(C) = \{(x, y) : (x, y) \in \mathbb{R}^2, (x - a)^2 + (y - b\cosh(r))^2 = b^2 \sinh^2(r)\}.$$

Theorem If, in a neutral geometry, $\overline{AB}$ is a chord of $\mathcal{C}_r(C)$ and ℓ is the perpendicular bisector of $\overline{AB}$, then $C \in \ell$.

Proof Since $AC = r = BC$, $C \in \ell$.

Theorem If, in a neutral geometry, $\mathcal{C}_r(S) \cap \mathcal{C}_s(D)$ has three or more points, then $C = D$ and $r = s$.

Proof Let P, Q, and R be three distinct points in $\mathcal{C}_r(S) \cap \mathcal{C}_s(D)$. Let ℓ be the perpendicular bisector of $\overline{PQ}$ and let m be the perpendicular bisector of $\overline{QR}$. Then $C \in \ell \cap m$ and $D \in \ell \cap m$. Hence either $C = D$, or C and D are distinct points and $\ell = m$.

Suppose $\ell = m$. Let M be the midpoint of $\overline{PQ}$ and N be the midpoint of $\overline{QR}$. If P, Q, and R were collinear, then

$$\ell \cap \overleftrightarrow{PQ} = \{M\} = \ell \cap \overleftrightarrow{QR} = \{N\}.$$

Hence $M = N$, which would imply $P = R$. Hence P, Q, and R must be noncollinear, in which case M, Q, and N are noncollinear. But then $\triangle MQN$ has two right angles, which is a contradiction. Hence $\ell \neq m$ and $C = D$.

Finally, we now have $r = PC = PD = s$.

Definition Given $\mathcal{C} = \mathcal{C}_r(C)$ in a metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$, we call

$$\text{int}(\mathcal{C}) = \{P : P \in \mathcal{P}, CP < r\}$$

the *interior* of $\mathcal{C}$ and we call

$$\text{ext}(\mathcal{C}) = \{P : P \in \mathcal{P}, CP > r\}$$

the *exterior* of $\mathcal{C}$.

Theorem The interior of a circle in a neutral geometry is convex.

Proof Let $\mathcal{C}$ be a circle with radius r and center C. Let $A, B \in \text{int}(\mathcal{C})$ and let $A - D - B$. If $C \in \overleftrightarrow{AB}$, then we may choose a ruler f for $\overleftrightarrow{AB}$ with $f(C) = 0$. Then $-r < f(A) < f(D) < f(B) < r$ or $-r < f(B) < f(D) < f(A) < r$, so $CD = |f(D)| < r$ and $D \in \text{int}(\mathcal{C})$.

Now suppose $C \notin \overleftrightarrow{AB}$. Then A, B, and C are noncollinear. It follows that CD is less than the larger of CA and CB. Hence $CD < r$, and $D \in \text{int}(\mathcal{C})$. Thus $\text{int}(\mathcal{C})$ is convex.

Theorem In a neutral geometry, a line intersects a circle in at most two points.

Proof See homework.

Definition Given a line ℓ and a circle $\mathcal{C}$ in a metric geometry, we say ℓ is *tangent* to $\mathcal{C}$ if $\ell \cap \mathcal{C}$ contains exactly one point and we call ℓ a *secant* of $\mathcal{C}$ if $\ell \cap \mathcal{C}$ contains exactly two points.

Theorem If, in a neutral geometry, $Q \in \mathcal{C}_r(C)$ and t is a line through Q, then t is tangent to $\mathcal{C}_r(C)$ if and only if $t \perp \overleftrightarrow{CQ}$.

Proof Suppose t is tangent to $\mathcal{C}_r(C)$ at Q. Let A be the foot of the perpendicular from C to t. If $A \neq Q$, then let $B \in t$ such that $Q - A - B$ and $\overline{AQ} \simeq \overline{AB}$. Then $\triangle CAB \simeq \triangle CAQ$ by Side-Angle-Side; in particular, $\overline{CB} \simeq \overline{CQ}$. Thus $CB = r$, and $r \in \mathcal{C}_r(C)$, contradicting the assumption that t is a tangent line. Hence $A = Q$ and $t \perp \overleftrightarrow{CA}$.

Now suppose $t \perp \overleftrightarrow{CQ}$. If $P \in t$, $P \neq Q$, then $\triangle CQP$ is a right triangle with hypotenuse CP. Hence $CP > CQ = r$, so $P \notin \mathcal{C}_r(C)$. Hence $t \cap \mathcal{C}_r(C) = \{Q\}$, so t is tangent to C.

Theorem Given any three points A, B, and C in a neutral geometry,

$$|AB - AC| \leq BC.$$

Proof From the Triangle Inequality, we have

$$AB \leq AC + CB,$$

from which we obtain

$$AB - AC \leq BC,$$

and

$$AC \leq AB + BC,$$

from which we obtain

$$AB - AC \geq -BC.$$

Hence $|AB - AC| \leq BC$.

Theorem If, in a neutral geometry $\{\mathcal{P}, \mathcal{L}, d, m\}$, $r > 0$ and A, B, and C are points such that $AC < r$ and $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$, then there exists a point $D \in \overrightarrow{AB}$ with $CD = r$.

Proof Let E be a point on $\overrightarrow{AB}$ with $AE = r$. Since CE is the hypotenuse of $\triangle CAE$, it follows that $CE > AE = r$.

Now let f be a ruler for $\overleftrightarrow{AB}$ with $f(A) = 0$ and $f(B) > 0$. Define $h : [0, r] \to \mathbb{R}$ by

$$h(t) = d(C, f^{-1}(t)).$$

Note that $h(0) = d(C, A) < r$ and $h(r) = d(C, E) > r$. If h is continuous, it follows there exists $s \in (0, r)$ such that $h(s) = r$. If we let $D = f^{-1}(s)$, then $r = h(s) = d(C, D)$.

It remains to show that h is continuous. Let $t_0 \in [0, r]$. Note that for any $t \in [0, r]$,

$$|h(t) - h(t_0)| = |d(C, f^{-1}(t)) - d(C, f^{-1}(t_0)| \leq d(f^{-1}(t), f^{-1}(t_0)) = |t - t_0|.$$

Hence given $\epsilon > 0$, if we let $\delta > 0$, then

$$|h(t) - h(t_0)| < \epsilon$$

whenever

$$|t - t_0| < \delta.$$

Thus h is continuous at t_0, and hence continuous on $[0, r]$.

Line-Circle Theorem If, in a neutral geometry, ℓ is a line, $\mathcal{C}$ is a circle, and $\ell \cap \text{int}(\mathcal{C}) \neq \emptyset$, then ℓ is a secant of $\mathcal{C}$.

Proof Suppose $\mathcal{C}$ has radius r and center C. If $C \in \ell$, let f be a ruler for ℓ with $f(C) = 0$. Then $f^{-1}(r)$ and $f^{-1}(-r)$ are both on ℓ and on $\mathcal{C}$. Hence ℓ is a secant line of $\mathcal{C}$.

Now suppose $C \notin \ell$. Let $P \in \ell \cap \text{int}(\mathcal{C})$. Then $CP < r$. Let A be the foot of the perpendicular from C to ℓ. If $A = P$, then $CA < r$. If $A \neq P$, then $\triangle CAP$ is a right triangle with hypotenuse CP. Hence $CA < CP < r$. If we let points B and D be points on

ℓ with $B - A - D$, then, by the previous theorem, there exist points $S \in \overrightarrow{AB}$ and $Q \in \overrightarrow{BD}$ with $CS = r$ and $CQ = r$. Hence ℓ is a secant line of $\mathcal{C}$.

External Tangent Theorem If, in a neutral geometry, $\mathcal{C}$ is a circle and $P \in \text{ext}(\mathcal{C})$, then there exist exactly two lines through P which are tangent to $\mathcal{C}$.

Proof Suppose $\mathcal{C}$ has center C and radius r. Since $P \in \text{ext}(\mathcal{C})$, $CP > r$, so there exists a unique point A with $C - A - P$ and $CA = r$. Let ℓ be the perpendicular to $\overleftrightarrow{CP}$ at A. Now $CA = r < CP$, so $A \in \text{int}(\mathcal{C}')$, where $\mathcal{C}'$ is the circle of radius CP with center at C. Hence, by the previous theorem, ℓ intersects $\mathcal{C}'$ at two points, say, Q and Q'.

Now $CQ = CP > r$, so there exists a unique point $B \in \overleftrightarrow{CQ}$ such that $C - B - Q$ and $CB = r$. Then $B \in \mathcal{C}$. Moreover, $\overline{PC} \simeq \overline{QC}$, $\angle PCB = \angle QCA$, and $\overline{CB} \simeq \overline{CA}$. Hence $\triangle PCB \simeq \triangle QCA$ by Side-Angle-Side. Thus $\angle PBC$ is a right angle since $\angle PBC \simeq \angle QAC$. Hence $\overleftrightarrow{PB} \perp \overleftrightarrow{CB}$, and so $\overleftrightarrow{PB}$ is tangent to $\mathcal{C}$ at B.

Using Q', we can construct another tangent $\overleftrightarrow{PB'}$, where B' is the unique point on CQ' with $C - B' - Q'$ and $CB' = r$. It is left to the homework to show that there are no other lines through P tangent to $\mathcal{C}$.