Lecture 21: Right Triangles

21.1 Right Triangles

Definition We call a triangle $\triangle ABC$ in a protractor geometry a *right triangle* if one of $\angle A$, $\angle B$, or $\angle C$ is a right angle. We call a side opposite a right angle in a right triangle a *hypotenuse*.

Definition In a protractor geometry, we call $\overline{AB}$ the *longest side* of $\triangle ABC$ if $\overline{AB} > \overline{AC}$ and $\overline{AB} > \overline{BC}$, and we call $\overline{AB}$ a *longest side* if $\overline{AB} \geq \overline{AC}$ and $\overline{AB} \geq \overline{BC}$.

Theorem If, in a neutral geometry, $\triangle ABC$ is a right triangle with $\angle C$ a right angle, then $\angle A$ and $\angle B$ are acute and $\overline{AB}$ is the longest side of $\triangle ABC$.

Proof Let D be a point such that $D - C - B$. Then $\angle DCA$ is a right angle, exterior to $\triangle ABC$ with remote interior angles $\angle A$ and $\angle B$. Hence $\angle DCA > \angle A$ and $\angle DCA > \angle B$. Since $\angle DCA \simeq \angle ACB$, it follows that $\angle C > \angle A$ and $\angle C > \angle B$. Hence $\angle A$ and $\angle B$ are both acute. Moreover, it also follows that $\overline{AB} > \overline{AC}$ and $\overline{AB} > \overline{BC}$.

Note that the previous result says that a right triangle has only one hypotenuse and that the hypotenuse is the longest side of a right triangle.

Definition Given a right triangle $\triangle ABC$ with right angle at C in a neutral geometry, we call $\overline{AC}$ and $\overline{BC}$ the *legs* of $\triangle ABC$.

Theorem If P and Q are points and ℓ is a line in a neutral geometry with $Q \in \ell$ and $P \notin L$, then $\overleftrightarrow{PQ} \perp \ell$ if and only if $PQ \leq PR$ for all $R \in \ell$.

Proof Suppose $\overleftrightarrow{PQ} \perp \ell$ and $R \in \ell$. If $R = Q$, then $PQ = PR$. If $R \neq Q$, then $\triangle PQR$ is a right triangle with hypotenuse PR. Hence $PQ < PR$. Thus, in either case, $PQ \leq PR$.

Now suppose $PQ \leq PR$ for all $R \in \ell$. Let S be the point on ℓ such that PS is the unique line through P which is perpendicular to ℓ. If $S \neq Q$, then $\triangle PQS$ is a right triangle with hypotenuse $\overline{PQ}$, from which it follows that $PS < PQ$, contradicting our assumption about Q. Hence we must have $S = Q$, and so $\overleftrightarrow{PQ} \perp \ell$.

Definition Given a line ℓ and a point P in a neutral geometry $\{\mathcal{P}, \mathcal{L}, d, m\}$, we call

$$d(P,\ell) = \begin{cases} d(P,Q), & \text{if } P \notin \ell \text{ and } Q \text{ is the unique point on } \ell \text{ for which } \overleftrightarrow{PQ} \perp \ell, \\ 0, & \text{if } P \in \ell, \end{cases}$$

the *distance* from P to ℓ.

Note that for any $P \in \ell$, $d(P,\ell) \leq d(P,Q)$ for all $Q \in \ell$, with $d(P,\ell) = d(P,Q)$ if and only if $\overleftrightarrow{PQ} \perp \ell$.

Definition Given $\triangle ABC$ in a neutral geometry with D being the unique point on $\overleftrightarrow{AB}$ for which $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$, we call $\overline{CD}$ the *altitude* from C and we call D the *foot* of the altitude from C.

Theorem If, in a neutral geometry, $\overline{AB}$ is the longest side of $\triangle ABC$ and D is the foot of the altitude from C, then $A - D - B$.

Proof Suppose $D - A - B$. Then $\overline{CB}$ is the hypotenuse of the right triangle $\triangle CBD$, and so $\overline{CB} > \overline{BD}$. But $\overline{DB} > \overline{AB}$, so $\overline{CB} > \overline{AB}$, contradicting the assumption that $\overline{AB}$ is the longest side of $\triangle ABC$.

Now suppose $D = A$. Then $\triangle ABC$ is right triangle with right angle at A, and so $\overline{BC} > \overline{AB}$, again contradicting the assumption that $\overline{AB}$ is the longest side of $\triangle ABC$.

Similarly, we cannot have $A - B - D$ nor $D = B$. Hence $A - D - B$.

Hypotenuse-Leg (HL) If, in a neutral geometry, $\triangle ABC$ and $\triangle DEF$ are right triangles with right angles at C and F, $\overline{AB} \simeq \overline{DE}$, and $\overline{AC} \simeq \overline{DF}$, then $\triangle ABC \simeq \triangle DEF$.

Note that we could prove Hypotenuse-Leg in the Euclidean Plane using the Pythagorean Theorem and Side-Side-Side.

Proof Let G be the point on $\overleftrightarrow{EF}$ such that $E - F - G$ and $\overline{FG} \simeq \overline{BC}$. Then $\overline{AC} \simeq \overline{DF}$, $\angle ACB \simeq \angle DFG$ (they are both right angles), and $\overline{CB} \simeq \overline{FG}$, and so $\triangle ACB \simeq \triangle DFG$ by Side-Angle-Side. In particular, $\overline{AB} \simeq \overline{DG}$, so $\overline{DG} \simeq \overline{DE}$. Hence $\triangle DEG$ is isosceles, and so $\angle DEF \simeq \angle DGF$. Thus $\triangle DEF \simeq \triangle DGF$ by Side-Angle-Angle. Hence $\triangle DEF \simeq \triangle ABC$.

Hypotenuse-Angle (HA) If, in a neutral geometry, $\triangle ABC$ and $\triangle DEF$ are right triangles with right angles at C and F, $\overline{AB} \simeq \overline{DE}$, and $\angle A \simeq \angle D$, then $\triangle ABC \simeq \triangle DEF$.

Proof See homework.

Theorem If, in a neutral geometry $\{\mathcal{P}, \mathcal{L}, d, m\}$, ℓ is the perpendicular bisector of $\overline{AB}$, then

$$\ell = \{P : P \in \mathcal{P}, AP = BP\}.$$

Proof Let P be a point with $AP = BP$. If $P \in \overleftrightarrow{AB}$, then P is the midpoint of $\overline{AB}$, and so $P \in \ell$. So suppose $P \notin \overleftrightarrow{AB}$. Let N be the unique point on $\overleftrightarrow{AB}$ such that $\overleftrightarrow{PN} \perp \overleftrightarrow{AB}$.

Note that $N \neq A$ since then $\triangle APB$ is a right triangle with hypotenuse $\overline{BP}$, which would imply that $BP > AP$. Similarly, $N \neq B$. Then $\overline{AP} \simeq \overline{BP}$ and $\overline{PN} \simeq \overline{PN}$ imply $\triangle PNA \simeq \triangle PNB$ by Hypotenuse-Leg. In particular, $\overline{NA} \simeq \overline{NB}$, from which it follows that N is the midpoint of $\overline{AB}$. Hence $\overleftrightarrow{PN} = \ell$, and so $P \in \ell$.

Now let $P \in \ell$. If $P \in \overleftrightarrow{AB}$, then P is the midpoint of $\overline{AB}$, and so $AP = BP$. If $P \notin \overleftrightarrow{AB}$, then $\triangle PMA \simeq \triangle PMB$, where M is the midpoint of $\overline{AB}$, by Side-Angle-Side. In particular, $AP = BP$. Thus $\ell = \{P : P \in \mathcal{P}, AP = BP\}$.

Theorem If, in a neutral geometry, $\overrightarrow{BD}$ is the bisector of $\angle ABC$, E is the foot of the perpendicular from D to $\overleftrightarrow{BA}$, and F is the foot of the perpendicular from D to $\overleftrightarrow{BC}$, then $\overline{DE} \simeq \overline{DF}$.

Proof See homework.