Lecture 20: Exterior Angle Theorem

20.1 Comparing segments and angles

Definition In a metric geometry, we say $\overline{AB}$ is *less than*, or *smaller than*, $\overline{CD}$, denoted $\overline{AB} < \overline{CD}$, if $AB < CD$, $\overline{AB}$ is *greater than*, or *larger than* $\overline{CD}$, denoted $\overline{AB} > \overline{CD}$, if $AB > CD$, and we write $\overline{AB} \leq \overline{CD}$ if either $\overline{AB} < \overline{CD}$ or $\overline{AB} \simeq \overline{CD}$.

Definition In a protractor geometry, we say $\angle ABC$ is *less than*, or *smaller than*, $\angle DEF$, denoted $\angle ABC < \angle DEF$, if $m(\angle ABC) < m(\angle DEF)$, $\angle ABC$ is *greater than*, or *larger than* $\angle DEF$, denoted $\angle ABC > \angle DEF$, if $m(\angle ABC) > m(\angle DEF)$, and we write $\angle ABC \leq \angle DEF$ if either $\angle ABC < \angle DEF$ or $\angle ABC \simeq \angle DEF$.

Theorem In a metric geometry, $\overline{AB} < \overline{CD}$ if and only if there exists a point $G \in \operatorname{int}(\overline{CD})$ such that $\overline{AB} \simeq \overline{CG}$.

Theorem In a protractor geometry, $\angle ABC < \angle DEF$ if and only if there exists a point $G \in \operatorname{int}(\angle DEF)$ such that $\angle ABC \simeq \angle DEG$.

20.2 Exterior angles

Definition Given $\triangle ABC$ in a protractor geometry and a point D with $A - C - D$, we call $\angle BCD$ an *exterior angle* of $\triangle ABC$ with *remote interior angles* $\angle BAC$ and $\angle ABC$.

Exterior Angle Theorem In a neutral geometry, an exterior angle of $\triangle ABC$ is greater than either of its remote interior angles.

Proof Given $\triangle ABC$, let D be a point with $A-C-D$. We will first show that $\angle BCD > \angle ABC$.

Let M be the midpoint of $\overline{AB}$ and let E be a point on $\overrightarrow{AM}$ with $A - M - E$ and $\overline{AM} \simeq \overline{ME}$. Then $\overline{AM} \simeq \overline{EM}$, $\angle AMC \simeq \angle EMC$ (they are vertical angles), and $\overline{MB} \simeq \overline{MC}$, and so $\triangle AMC \simeq \triangle EMC$ by Side-Angle-Side. In particular, $\angle ABC \simeq \angle BCE$. Since $E \in \operatorname{int}(\angle BCD)$ (this was a homework problem), it follows that

$$\angle BCD > \angle BCE \simeq \angle ABC,$$

and so $\angle BCD > \angle ABC$.

It remains to show that $\angle BCD > \angle BAC$. If we let D' be a point on $\overrightarrow{BC}$ with $B - C - D'$, then, by what we have already shown, $\angle ACD' > \angle BAC$. Since $\angle ACD' \simeq \angle BCD$ (see homework), it follows that $\angle BCD > \angle BAC$.

Theorem Given a point P and a line ℓ in a neutral geometry, there exists a unique line through P perpendicular to ℓ.

Proof We have already proven the result for $P \in \ell$. If $P \notin \ell$, we have already shown there exists at least one line through P perpendicular to ℓ.

Now suppose lines m and n are both perpendicular to ℓ. Let $\{A\} = \ell \cap m$, $\{B\} = \ell \cap n$, and let C be a point on ℓ with $A - B - C$. Then $\angle PBC$ is an exterior angle of $\triangle APB$, and so $\angle PBC > \angle PAB$, contradicting the assumption that both $\angle PAB$ and $\angle PBC$ are right angles (and hence congruent).

Side-Angle-Angle (SAA) If, in a neutral geometry, triangles $\triangle ABC$ and $\triangle DEF$ are such that $\overline{AB} \simeq \overline{DE}$, $\angle A \simeq \angle D$, and $\angle C \simeq \angle F$, then $\triangle ABC \simeq \triangle DEF$.

Proof If $\overline{AC} \simeq \overline{DF}$, then $\triangle BAC \simeq \triangle EDF$ by Side-Angle-Side. Thus $\triangle ABC \simeq \triangle DEF$.

So suppose $AC < DF$. Let $G \in \overline{DF}$ so that $\overline{AC} \simeq DG$. Then $\triangle BAC \simeq \triangle EDG$ by Side-Angle-Side. In particular, $\angle ACB \simeq \angle DGE$. However, $\angle DGE$ is an exterior angle of $\triangle EGF$; in particular, $\angle DGE > \angle GFE = \angle DFE$. Thus we have $\angle ACB > \angle DFE$, contradicting our assumptions. Thus we cannot have $AC < DF$. Similarly, we cannot have $DF < AC$, and so $\overline{AC} \simeq \overline{DF}$, and $\triangle ABC \simeq \triangle DEF$.

Note: Another approach to Side-Angle-Angle would be to argue that $\angle B \simeq \angle E$ and then conclude $\triangle ABC \simeq \triangle DEF$ by Side-Angle-Side. Why can't we use this approach?

Theorem Given $\triangle ABC$ in a neutral geometry, if $\overline{AB} > \overline{AC}$, then $\angle C > \angle B$.

Proof Let $D \in \overrightarrow{AC}$ be such that $A - C - D$ and $\overline{AD} \simeq \overline{AB}$. Then $\triangle ABC$ is isosceles, and so $\angle ADB \simeq \angle ABD$. Moreover, $C \in \text{int}(\angle ABD)$, so $\angle ABD > \angle ABC$. Since $\angle ACB$ is an exterior angle of $\triangle BCD$, we now have

$$\angle ABC < \angle ABD \simeq \angle ADB < \angle ACB.$$

Theorem Given $\triangle ABC$ in a neutral geometry, if $\angle C > \angle B$, then $\overline{AB} > \overline{AC}$.

Proof See homework.

Triangle Inequality Given $\triangle ABC$ in a neutral geometry, $AC < AB + BC$.

Proof Let $D \in \overrightarrow{BC}$ such that $C - B - D$ and $\overline{BD} \simeq \overline{AB}$. Then $\triangle ABD$ is isosceles, so $\angle BAD \simeq \angle ADB$. Now $B \in \text{int}(\angle DAC)$, so $\angle BAD < \angle CAD$. Hence, in $\triangle ACD$, $\angle ADC < \angle CAD$, and so

$$AC < DC = CB + BD = BC + AB.$$

Open Mouth, or Hinge, Theorem Given $\triangle ABC$ and $\triangle DEF$, in a neutral geometry, with $\overline{AB} \simeq \overline{DE}$, $\overline{BC} \simeq \overline{EF}$, and $\angle B > \angle E$, then $\overline{AC} > \overline{DF}$.

Proof Let H be the point on the same side of $\overleftrightarrow{BC}$ as A such that $\angle HBC \simeq \angle DEF$ and $\overline{BH} \simeq \overline{ED}$. Then $\triangle DEF \simeq \triangle HBC$ by Side-Angle-Side. Note that $H \in \mathrm{int}(\angle ABC)$ since $\angle DEF < \angle ABC$. Hence $\overrightarrow{BH}$ intersects $\overline{AC}$ at a single point, say, K.

Let M be the point on $\overline{AK}$ at which the angle bisector of $\angle KBA$ intersects $\overline{AK}$. Note that $\triangle ABM \simeq \triangle HBM$ by Side-Angle-Side; in particular, $\overline{AM} \simeq \overline{HM}$. Now if $H \neq K$, C, H, and M are noncollinear, and, applying the Triangle Inequality to $\triangle CHM$, we have

$$HC < HM + MC.$$

If $K = H$, then $C - H - M$; hence $MC > HC$, and so, again,

$$HC < HM + MC.$$

Thus, in either case,

$$HC < HM + MC = AM + MC = AC.$$

Since $\overline{HC} \simeq \overline{DF}$, we have $\overline{AC} > \overline{DF}$.

Theorem If, in a neutral geometry, $\triangle ABC$ is such that $\overline{AB} \leq \overline{CB}$ and D is a point such that $A - D - C$, then $\overline{DB} < \overline{CB}$.

Proof See homework.