Lecture 2: Sets and Functions

2.1 Sets

We call a collection of objects a *set*. Technically, a set is an undefined object satisfying certain axioms, but this loose definition is sufficient for our purposes. We will insist that a set be defined using a well-defined property which determines whether a particular object is in the set or not.

Notation:

- If S is a set, we write $x \in S$ to indicate that x is an element of S and $T \subset S$ to indicate that T is a subset of S (that is, if $x \in T$, then $x \in S$). Of course, $x \notin S$ means x is not an element of S. Note that $S = T$ if and only if $S \subset T$ and $T \subset S$.

- If A and B are sets, then

  $$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

  is the *union* of A and B,

  $$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

  is the *intersection* of A and B, and

  $$A - B = \{x : x \in A \text{ and } x \notin B\}$$

  is the *difference* of A and B.

- $\emptyset$ is the *empty set*, that is, the set with no elements. Note that given any set S, $\emptyset \subset S$.

- If $A \cap B = \emptyset$, we say A and B are *disjoint*.

- If A and B are sets, then

  $$A \times B = \{(a, b) : a \in A, b \in B\}$$

  is the *Cartesian product* of A and B.

Example If $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, then

$$A \cup B = \{1, 2, 3, 4\},$$

$$A \cap B = \{2\},$$

$$A - B = \{1\},$$

$$B - A = \{3, 4\},$$

and

$$A \times B = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4)\}.$$

Example We will show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

We first show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$: Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \int B \cup C$. If $x \in B$, then $x \in A \cap B$; if $x \in C$, then $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$, so $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

We now show that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$: Suppose $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, so $x \in A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C$, so $x \in A \cap (B \cup C)$. Hence $x \in A \cap (B \cup C)$, so $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Example We let $\mathbb{R}$ represent the set of all real numbers and $\mathbb{Z}$ represent the set of all integers. $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the *Cartesian plane*, that is,

$$\mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}.$$

Similarly,

$$\mathbb{Z} \times \mathbb{Z} = \{(m, n) : m \in \mathbb{Z}, n \in \mathbb{Z}\}.$$

2.2 Equivalence relations

Definition Given a set S, a *binary relation* on S is a subset of $S \times S$.

Note: If $R \subset S \times S$ is a binary relation and $(x, y) \in R$, then we say x and y are *related* and write, generically, $x \sim y$ (for certain specific binary relations we may use a symbol other than $\sim$).

Example If $T = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, x < y\}$, then T is a binary relation on $\mathbb{R}$.

Example If $E = \{(a, b) : a \in \mathbb{Z}, y \in Z, a - b \text{ is divisible by } 2\}$, then E is a binary relation on $\mathbb{Z}$.

Definition We say a binary relation on a set S is *reflexive* if for every $a \in S$, $a \sim a$; *symmetric* if for every $a \in S$ and $b \in S$, $a \sim b$ implies $b \sim a$; and *transitive* if for every $a \in S$, $b \in S$, and $c \in S$, $a \sim b$ and $b \sim c$ imply $a \sim c$. A binary relation is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example The binary relation $T = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, x < y\}$ is transitive, but neither reflexive nor symmetric.

Example The binary relation $E = \{(a,b) : a \in \mathbb{Z}, y \in Z, a - b \text{ is divisible by } 2\}$ is an equivalence relation: It is reflexive since for any $a \in S$,

$$a - a = 2 \cdot 0;$$

symmetric since if $a \sim b$, then $a - b = 2k$ for some integer k, from which it follows that $b - a = 2(-k)$; and transitive since if $a \sim b$ and $b \sim c$, then $a - b = 2k$ for some integer k and $b - c = 2m$ for some integer m, and so

$$a - c = (a - b) + (b - c) = 2k + 2m = 2(k + m).$$

Note: More generally, for integers a, b, and n, we write $a \equiv b(n)$ to indicate that $a - b$ is divisible by n, in which case we say that a is *equivalent to* b *modulo* n.

Definition Given an equivalence relation on a set S, the *equivalence class* of $s \in S$ is the set

$$[s] = \{x : x \in S, x \sim s\}.$$

Example For integers a and b, let $a \sim b$ if $a \equiv b(2)$. Then

$$[1] = \{x : x \in \mathbb{Z}; x \text{ is odd}\} = [3] = [5] = \cdots$$

and

$$[0] = \{x : x \in \mathbb{Z}; x \text{ is even}\} = [2] = [4] = \cdots.$$

Theorem Given an equivalence relation on S, if $a \in S$ and $b \in S$, then either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Proof Suppose $[a] \cap [b] \neq \emptyset$. Let $x \in [a]$. Since $[a] \cap [b] \neq \emptyset$, there exists a z such that $a \sim z$ and $b \sim z$. Since $x \sim a$, it follows that $x \sim z$. Hence $x \sim z$ and $z \sim b$, so $x \sim b$. Thus $x \in [b]$ and $[a] \subset [b]$. Similarly, if $x \in [b]$, then $x \sim b$, $b \sim z$, and $z \sim a$, so $x \sim a$. Thus $x \in [a]$, and so $[b] \subset [a]$. Hence $[a] = [b]$.

2.3 Functions

Definition If $f : S \to T$ is a function, then the *image* of f is the set

$$\text{Im}(f) = \{t : t \in T, t = f(s) \text{ for some } s \in S\}.$$

We say f is *surjective* if $\text{Im}(f) = T$. We say f is *injective* if for every a and b in S, $f(a) = f(b)$ implies $a = b$. We say f is a *bijection*, or a *one-to-one correspondence*, if it is both surjective and injective.

Example $f : \mathbb{R} \to (0, \infty)$ given by $f(t) = e^t$ is a bijection. It is surjective because if $s \in (0, \infty)$, then $s = e^t$ where $t = \log(s)$; it is injective because if $f(a) = f(b)$, then $e^a = e^b$, from which it follows that $a = b$.

Example $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{1 + x^2}$$

is neither injective nor surjective. It is not injective because, for example, $f(-2) = f(2)$; it is not surjective because, for example, $0 \notin \mathrm{Im}(f)$.

Theorem If $f : S \to T$ and $g : T \to V$ are both surjections, then $g \circ f : S \to V$ is also a surjection.

Proof If $v \in V$, then, since g is surjective, there exists a $t \in T$ for which $g(t) = v$. Now since f is surjective, there exists an $s \in S$ for which $f(s) = t$. Then

$$g \circ f(s) = g(f(s)) = g(t) = v,$$

so $v \in \mathrm{Im}(g \circ f)$. Hence $g \circ f$ is a surjection.

Notation: Given a set S, we let $\mathrm{id}_S : S \to S$ denote the *identity function*, that is, the function defined by $\mathrm{id}_S(s) = s$ for all $s \in S$.

Definition If $f : S \to T$ is a bijection, then we call the function $g : T \to S$ defined by $g(t) = s$, where s is the unique element of S with the property that $f(s) = t$, the *inverse* of f, which we denote f^{-1}.

Theorem A function $f : S \to T$ is a bijection if and only if there exists a function $g : T \to S$ such that $g \circ f = \mathrm{id}_S$ and $f \circ g = \mathrm{id}_T$.

Proof If f is a bijection, then let $g = f^{-1}$.

Now suppose there exists a function $g : T \to S$ such that $g \circ f = \mathrm{id}_S$ and $f \circ g = \mathrm{id}_T$. Given $t \in T$, let $s = g(t)$. Then $f(s) = f(g(t)) = t$, so f is surjective. Suppose $f(a) = f(b)$. Then $g(f(a) = g(f(b))$, and so $a = b$. Thus f is injective. Hence f is a bijection.