Lecture 19: Congruence Theorems

19.1 Congruence theorems

Definition A protractor geometry satisfies the *Angle-Side-Angle Axiom* (ASA) if, given $\triangle ABC$ and $\triangle DEF$, $\angle A \simeq \angle D$, $\overline{AB} \simeq \overline{DE}$, and $\angle B \simeq \angle E$ imply $\triangle ABC \simeq \triangle DEF$.

Theorem A neutral geometry satisfies Angle-Side-Angle.

Proof Given $\triangle ABC$ and $\triangle DEF$ with $\angle A \simeq \angle D$, $\overline{AB} \simeq \overline{DE}$, and $\angle B \simeq \angle E$, let G be a point on $\overrightarrow{DF}$ with $\overline{DG} \simeq \overline{AC}$. Then $\triangle BAC \simeq \triangle EDG$ by Side-Angle-Side. In particular, $\angle ABC \simeq \angle DEG$. But $\angle ABC \simeq \angle DEF$, so $\angle DEG \simeq \angle DEF$. Now G and F are on the same side of $\overleftrightarrow{DE}$ (since $G \in \overrightarrow{DF}$), it follows that $\overrightarrow{EF} = \overrightarrow{EG}$. Thus

$$\{F\} = \overrightarrow{EF} \cap \overrightarrow{DF} = \overrightarrow{EG} \cap \overrightarrow{DF} = \{G\},$$

so $G = F$ and $\triangle BAC \simeq \triangle EDF$. Hence $\triangle ABC \simeq \triangle DEF$.

Theorem Given $\triangle ABC$ in a neutral geometry with $\angle A \simeq \angle C$, then $\overline{AB} \simeq \overline{CB}$. In particular, $\triangle ABC$ is isosceles.

Proof See homework.

Definition A protractor geometry satisfies the *Side-Side-Side Axiom* (ASA) if, given $\triangle ABC$ and $\triangle DEF$, $\overline{AB} \simeq \overline{DE}$, $\overline{BC} \simeq \overline{EF}$, and $\overline{CA} \simeq \overline{FD}$ imply $\triangle ABC \simeq \triangle DEF$.

Theorem A neutral geometry satisfies Side-Side-Side.

Proof Given $\triangle ABC$ and $\triangle DEF$ with $\overline{AB} \simeq \overline{DE}$, $\overline{BC} \simeq \overline{EF}$, and $\overline{CA} \simeq \overline{FD}$, let B' be a point on the opposite side of $\overleftrightarrow{AC}$ as B with $m(\angle CAB') = m(\angle FDE)$ and $\overline{AB'} \simeq \overline{DE}$ (and so $\overline{AB'} \simeq \overline{AB}$). Then $\triangle CAB' \simeq \triangle FDE$ by Side-Angle-Side. Hence we need show only that $\triangle AB'C \simeq \triangle ABC$.

Let G be the such that $\overline{BB'} \cap \overleftrightarrow{AC} = \{G\}$. Then $G - A - C$, $G = A$, $A - G - C$, $G = C$, or $A - C - G$. We will consider the first case, the second and third cases are left to the homework, the fourth case is the same as the second case, and the fifth case is the same as the first case. So suppose $G - A - C$. Then $\triangle BAB'$ is isosceles ($\overline{AB} \simeq \overline{AB'}$), so $\angle ABB' \simeq \angle AB'B$. Moreover, $\triangle BCB'$ is isosceles (since $\overline{B'C} \simeq \overline{EF} \simeq \overline{BC}$), so $\angle CBB' \simeq \angle CB'B$. Since $A \in \text{int}\angle CBB'$ and $A \in \text{int}\angle CBB'$, it follows, from the Angle Subtraction Theorem, that $\angle CBA \simeq \angle CB'A$.

Hence we now have $\overline{AB} \simeq \overline{AB'}$, $\angle ABC \simeq \angle AB'C$, and $\overline{BC} \simeq \overline{B'C}$. Thus $\triangle ABC \simeq \triangle AB'C$, and so $\triangle ABC \simeq \triangle DEF$.

Theorem If a protractor geometry satisfies Angle-Side-Angle, then it satisfies Side-Angle-Side.

Proof See homework

Theorem Given a line ℓ and a point P not on ℓ in a neutral geometry, then there exists at least one line through P perpendicular to ℓ.

Proof Let A and C be distinct points on ℓ. Let Q be the unique point on the opposite side of $\overleftrightarrow{AC}$ as P such that $\angle CAQ \simeq \angle CAP$ and $\overline{AQ} \simeq \overline{AP}$. Let G be the point such that $\overline{PQ} \cap \overleftrightarrow{AC} = \{G\}$.

If $G = A$, the $\angle CAQ$ and $\angle CAP$ form a linear pair; since they are congruent, we must have

$$m(\angle CAQ) = m(\angle CAP) = 90.$$

Hence $\overleftrightarrow{PQ} \perp \ell$.

If $G \neq A$, then $\triangle PAG \simeq \triangle QAG$ by Side-Angle-Side. Then $\angle AGP \simeq \angle AGQ$; since $\angle AGP$ and $\angle AGQ$ form a linear pair, it follows that

$$m(\angle AGP) = m(\angle AGQ) = 90.$$

Hence $\overleftrightarrow{PQ} \perp \ell$.