Lecture 18: Side-Angle-Side

18.1 Congruence

Notation: Given $\triangle ABC$, we will write $\angle A$ for $\angle BAC$, $\angle B$ for $\angle ABC$, and $\angle C$ for $\angle BCA$ if the meaning is clear from the context.

Definition In a protractor geometry, we write $\triangle ABC \simeq \triangle DEF$ if

$$\overline{AB} \simeq \overline{DE}, \qquad \overline{BC} \simeq \overline{EF}, \quad \overline{CA} \simeq \overline{FD},$$

and

$$\angle A \simeq \angle D, \qquad \angle B \simeq \angle E, \qquad \angle C \simeq \angle F.$$

Definition In a protractor geometry, if $\triangle ABC \simeq \triangle DEF$, $\triangle ACB \simeq \triangle DEF$, $\triangle BAC \simeq \triangle DEF$, $\triangle BCA \simeq \triangle DEF$, $\triangle CAB \simeq \triangle DEF$, or $\triangle CBA \simeq \triangle DEF$, we say $\triangle ABC$ and $\triangle DEF$ are *congruent*

Example In the Taxicab Plane, let $A = (0,0)$, $B = (-1,1)$, $C = (1,1)$, $D = (5,0)$, $E = (5,2)$, and $F = (7,0)$. Then

$$AB = 2 = DE,$$

$$AC = 2 = DF,$$

$$m(\angle A) = 90 = m(\angle D),$$

$$m(\angle B) = 45 = m(\angle E),$$

and

$$m(\angle C) = 45 = m(\angle F),$$

and so $\overline{AB} \simeq \overline{DE}$, $\overline{AC} \simeq \overline{DF}$, $\angle A \simeq \angle D$, $\angle B \simeq \angle E$, and $\angle C \simeq \angle F$. However,

$$BC = 2 \neq 4 = EF.$$

Thus $\overline{BC}$ and $\overline{EF}$ are not congruent, and so $\triangle ABC$ and $\triangle DEF$ are not congruent.

18.2 Side-angle-side

Definition A protractor geometry satisfies the *Side-Angle-Side Axiom* (SAS) if, given $\triangle ABC$ and $\triangle DEF$, $\overline{AB} \simeq \overline{DE}$, $\angle B \simeq \angle E$, and $\overline{BC} \simeq \overline{EF}$ imply $\triangle ABC \simeq \triangle DEF$.

Definition We call a protractor geometry satisfying the Side-Angle-Side Axiom a *neutral geometry*, also called an *absolute geometry*.

Example We will show that the Euclidean Plane is a neutral geometry. First recall the Law of Cosines: Given any triangle $\triangle ABC$ in the Euclidean Plane,

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos(m_E(\angle B)).$$

Note that, in particular, the measure of any angle of a triangle in the Euclidean Plane is determined by the lengths of the sides of the triangle. Hence given $\triangle ABC$ and $\triangle DEF$ with $\overline{AB} \simeq \overline{DE}$, $\angle B \simeq E$, and $\overline{BC} \simeq \overline{EF}$, we need show only that $\overline{AC} \simeq \overline{DF}$ to conclude that $\triangle ABC \simeq \triangle DEF$. Now

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos(m_E(\angle B)) \\ &= DE^2 + EF^2 - 2(DE)(EF)\cos(m_E(\angle E)) \\ &= DF^2, \end{aligned}$$

so $AC = DF$. Thus $\overline{AC} \simeq \overline{DF}$.

Example The Poincaré Plane is a neutral geometry. We will omit the proof, which is more easily done with the help of an axiom about isometries of the plane which is equivalent to the Side-Angle-Side Axiom.

18.3 Isosceles triangles

Definition In a protractor geometry, we say a triangle with two congruent sides is *isosceles*. We say a triangle which is not isosceles is *scalene*. It $\triangle ABC$ is isosceles with $\overline{AB} \simeq \overline{BC}$, then we call $\angle A$ and $\angle B$ the *base angles* of $\triangle ABC$. We say a triangle with all three sides congruent is *equilateral*.

Pons Asinorum In a neutral geometry, the base angles of an isosceles triangle are congruent.

Proof Consider a triangle $\triangle ABC$ with $\overline{AB} \simeq \overline{BC}$. Then $\overline{AB} \simeq \overline{CB}$, $\angle ABC \simeq \angle CBA$, and $\overline{BC} \simeq \overline{BA}$. Hence, by Side-Angle-Side, $\triangle ABC \simeq \triangle CBA$. In particular, $\angle BAC \simeq \angle BCA$.