Lecture 17: Complementary and Supplementary Angles

17.1 Complementary and supplementary angles

**Definition**  In a protractor geometry \(\{P, L, d, m\}\), we say \(\angle ABC\) is an *acute angle* if \(m(\angle ABC) < 90\), a *right angle* if \(m(\angle ABC) = 90\), and an *obtuse angle* if \(m(\angle ABC) > 90\).

We say angles \(\angle ABC\) and \(\angle DEF\) are *supplementary* if
\[
m(\angle ABC) + m(\angle DEF) = 180
\]
and we say angles \(\angle ABC\) and \(\angle DEF\) are *complementary* if
\[
m(\angle ABC) + m(\angle DEF) = 90.
\]

We say \(\angle ABC\) and \(\angle CBD\) form a *linear pair* if \(A - B - D\) and \(\angle ABC\) and \(\angle DBF\) form a *vertical pair* if either \(A - B - D\) and \(C - B - F\) or \(A - B - F\) and \(C - B - D\).

**Theorem**  Suppose \(C\) and \(D\) are points in a protractor geometry on the same side of \(\overrightarrow{AB}\). If \(m(\angle ABC) < m(\angle ABD)\), then \(C \in \text{int}(\angle ABD)\).

**Proof**  We need only show that \(C\) and \(A\) are on the same side of \(\overrightarrow{BD}\). Note first that \(C \notin \overrightarrow{BD}\) since, if \(C \in \overrightarrow{BD}\), then \(C \in \overrightarrow{BD}\) and \(\angle ABD = \angle ABC\), from which it would follow that \(m(\angle ABD) = m(\angle ABC)\).

Suppose \(C\) and \(A\) are on opposite sides of \(\overrightarrow{BD}\). Then \(D \in \text{int}(\angle ABC)\), so
\[
m(\angle ABC) = m(\angle ABD) + m(\angle DBC) > m(\angle ABD),
\]
contradicting our assumption that \(m(\angle ABC) < m(\angle ABD)\). Hence we must have \(C\) and \(A\) on the same side of \(\overrightarrow{BD}\), and so \(C \in \text{int}(\angle ABD)\).

**Linear Pair Theorem**  In a protractor geometry, angles which form a linear pair are supplementary.

**Proof**  Suppose \(\angle ABC\) and \(\angle CBD\) form a linear pair. Let
\[
\alpha = m(\angle ABC)
\]
and
\[
\beta = m(\angle CBD).
\]
We need to show that \(\alpha + \beta = 180\).
Suppose $\alpha + \beta < 180$. Then there exists a ray $\vec{BE}$ with $E$ on the same side of $\vec{AB}$ as $C$ such that $m(\angle ABE) = \alpha + \beta$. Now $C \in \text{int}(\angle ABE)$ (since $m(\angle ABC) < m(\angle ABE)$), so
\[
\alpha + \beta = m(\angle ABE) = m(\angle ABC) + m(\angle CBE) = \alpha + m(\angle CBE).
\]
Hence $m(\angle CBE) = \beta$. Moreover, $E \in \text{int}(\angle CBD)$, so
\[
\beta = m(\angle CBD) = m(\angle CBE) + m(\angle EBD) = \beta + m(\angle EBD),
\]
which implies $m(\angle EBD) = 0$, a contradiction. Hence $\alpha + \beta \geq 180$.

Suppose $\alpha + \beta > 180$. Then there exists a ray $\vec{BF}$ with $F$ on the same side of $\vec{AB}$ as $C$ such that $m(\angle ABF) = \alpha + \beta - 180$. Now
\[
\alpha + \beta - 180 = \alpha - (180 - \beta) \lt \alpha
\]
since $180 - \beta > 0$. Hence $F \in \text{int}(\angle ABC)$. Thus
\[
\alpha = m(\angle ABC) = m(\angle ABF) + m(\angle CBF) = \alpha + \beta - 180 + m(\angle CBF),
\]
so $m(\angle CBF) = 180 - \beta$. Moreover, $C \in \text{int}(\angle FBD)$, so
\[
m(\angle FBD) = m(\angle FBC) + m(\angle CBD) = (180 - \beta) + \beta = 180,
\]
which is a contradiction. Hence $\alpha + \beta = 180$.

**Theorem**  If, in a protractor geometry,
\[
m(\angle ABC) + m(\angle CBD) = m(\angle ABD),
\]
then $C \in \text{int}(\angle ABD)$.

**Proof**  Since
\[
m(\angle ABC) < m(\angle ABD),
\]
it follows from an earlier result that we need only show that $C$ and $D$ are on the same side of $\vec{AB}$. Suppose $C$ and $D$ lie on opposite sides of $\vec{AB}$. Now if $A$ and $D$ are on the same side of $\vec{BC}$, then $A \in \text{int}(\angle CBD)$. Hence
\[
m(\angle CBD) = m(\angle ABC) + m(\angle ABD).
\]
But $m(\angle ABD) > m(\angle CBD)$, so this is a contradiction.

If $A$ and $D$ are on the opposite sides of $\vec{BC}$, let $E$ be a point such that $E - B - A$. Then $E$ and $D$ are on the same side of $\vec{BC}$, and so
\[
m(\angle CBE) + m(\angle EBD) = m(\angle CBD).
Now $\angle ABC$ and $\angle CBE$ are a linear pair, so

$$m(\angle CBE) = 180 - m(\angle ABC).$$

Hence

$$180 - m(\angle ABC) + m(\angle EBD) = m(\angle CBD),$$

and so

$$180 + m(\angle EBD) = m(\angle ABC) + m(\angle CBD) = m(\angle ABD).$$

But this would mean $m(\angle ABD) > 180$, a contradiction. Hence we must have $C$ and $D$ on the same side of $\overline{AB}$.

**Theorem** If, in a protractor geometry, $A$ and $D$ lie on opposite sides of $\overline{BC}$ and $m(\angle ABC) + m(\angle CBD) = 180$, then $A - B - D$, that is, $\angle ABC$ and $\angle CBD$ form a linear pair.

**Proof** See homework.

### 17.2 Perpendicular lines

**Definition** We say lines $\ell$ and $m$ in a protractor geometry are *perpendicular*, denoted $\ell \perp m$, if $\ell \cup m$ contains a right angle. We say two rays or segments are *perpendicular* if the lines they determine are perpendicular.

**Theorem** Given a line $\ell$ and a point $P \in \ell$ in a protractor geometry, there exists a unique line $m$ through $P$ which is perpendicular to $\ell$.

**Proof** See homework.

**Theorem** Given a segment $\overline{AB}$ in a protractor geometry, there exists a unique line $\ell$ such that $\ell \perp \overline{AB}$ and $\ell \cap \overline{AB} = \{M\}$ where $AM = MB$.

**Proof** See homework.

We call the point $M$ in the previous theorem the *midpoint* of $\overline{AB}$ and we call $\ell$ the *perpendicular bisector* of $\overline{AB}$.

**Theorem** Given an angle $\angle ABC$ in a protractor geometry, there exists a unique ray $\overrightarrow{BD}$ with $D \in \text{int}(\angle ABC)$ and $m(\angle ABD) = m(\angle DBC)$.

**Proof** See homework.

We call $\overrightarrow{BD}$ in the previous theorem the *angle bisector* of $\angle ABC$. 
17.3 Angle congruence

**Definition** In a protractor geometry, we say angles $\angle ABC$ and $\angle DEF$ are congruent, denoted

$$\angle ABC \simeq \angle DEF,$$

if $m(\angle ABC) = m(\angle DEF)$.

**Vertical Angle Theorem** If, in a protractor geometry, $\angle ABC$ and $\angle DBF$ form a vertical pair, then $\angle ABC \simeq \angle DBF$.

**Proof** See homework.

**Angle Construction Theorem** Given $\angle ABC$ and a ray $\overrightarrow{ED}$ which lies on the edge of a half plane $H$ in a protractor geometry, then there exists a unique ray $\overrightarrow{EF}$ with $F \in H$ such that $\angle DEF \simeq \angle ABC$.

**Proof** See homework.

**Angle Addition Theorem** If, in a protractor geometry, $D \in \text{int}(\angle ABC)$, $S \in \text{int}(PQR)$, $\angle ABD \simeq \angle PQS$, and $\angle DBC \simeq \angle SQR$, then $\angle ABC \simeq \angle PQR$.

**Proof** See homework.

**Angle Subtraction Theorem** If, in a protractor geometry, $D \in \text{int}(\angle ABC)$, $S \in \text{int}(PQR)$, $\angle ABD \simeq \angle PQS$, and $\angle ABC \simeq \angle PQR$, then $\angle DBC \simeq \angle SQR$. 