Lecture 17: Complementary and Supplementary Angles

17.1 Complementary and supplementary angles

Definition In a protractor geometry $\{\mathcal{P}, \mathcal{L}, d, m\}$, we say $\angle ABC$ is an *acute angle* if $m(\angle ABC) < 90$, a *right angle* if $m(\angle ABC) = 90$, and an *obtuse angle* if $m(\angle ABC) > 90$. We say angles $\angle ABC$ and $\angle DEF$ are *supplementary* if

$$m(\angle ABC) + m(\angle DEF) = 180$$

and we say angles $\angle ABC$ and $\angle DEF$ are *complementary* if

$$m(\angle ABC) + m(\angle DEF) = 90.$$

We say $\angle ABC$ and $\angle CBD$ form a *linear pair* if $A - B - D$ and $\angle ABC$ and $\angle DBF$ form a *vertical pair* if either $A - B - D$ and $C - B - F$ or $A - B - F$ and $C - B - D$.

Theorem Suppose C and D are points in a protractor geometry on the same side of $\overleftrightarrow{AB}$. If $m(\angle ABC) < m(\angle ABD)$, then $C \in \text{int}(\angle ABD)$.

Proof We need only show that C and A are on the same side of $\overleftrightarrow{BD}$. Note first that $C \notin \overleftrightarrow{BD}$ since, if $C \in \overleftrightarrow{BD}$, then $C \in \overrightarrow{BD}$ and $\angle ABD = \angle ABC$, from which it would follow that $m(\angle ABD) = m(\angle ABC)$.

Suppose C and A are on opposite sides of $\overleftrightarrow{BD}$. Then $D \in \text{int}(\angle ABC)$, so

$$m(\angle ABC) = m(\angle ABD) + m(\angle DBC) > m(\angle ABD),$$

contradicting our assumption that $m(\angle ABC) < m(\angle ABD)$. Hence we must have C and A on the same side of $\overleftrightarrow{BD}$, and so $C \in \text{int}(\angle ABD)$.

Linear Pair Theorem In a protractor geometry, angles which form a linear pair are supplementary.

Proof Suppose $\angle ABC$ and $\angle CBD$ form a linear pair. Let

$$\alpha = m(\angle ABC)$$

and

$$\beta = m(\angle CBD).$$

We need to show that $\alpha + \beta = 180$.

Suppose $\alpha + \beta < 180$. Then there exists a ray $\overrightarrow{BE}$ with E on the same side of $\overleftrightarrow{AB}$ as C such that $m(\angle ABE) = \alpha + \beta$. Now $C \in \text{int}(\angle ABE)$ (since $m(\angle ABC) < m(\angle ABE)$), so

$$\alpha + \beta = m(\angle ABE) = m(\angle ABC) + m(\angle CBE) = \alpha + m(\angle CBE).$$

Hence $m(\angle CBE) = \beta$. Moreover, $E \in \text{int}(\angle CBD)$, so

$$\beta = m(\angle CBD) = m(\angle CBE) + m(\angle EBD) = \beta + m(\angle EBD),$$

which implies $m(\angle EBD) = 0$, a contradiction. Hence $\alpha + \beta \geq 180$.

Suppose $\alpha + \beta > 180$. Then there exists a ray $\overrightarrow{BF}$ with F on the same side of $\overleftrightarrow{AB}$ as C such that $m(\angle ABF) = \alpha + \beta - 180$. Now

$$\alpha + \beta - 180 = \alpha - (180 - \beta) < \alpha$$

since $180 - \beta > 0$. Hence $F \in \text{int}(\angle ABC)$. Thus

$$\alpha = m(\angle ABC) = m(\angle ABF) + m(\angle CBF) = \alpha + \beta - 180 + m(\angle CBF),$$

so $m(\angle CBF) = 180 - \beta$. Moreover, $C \in \text{int}(\angle FBD)$, so

$$m(\angle FBD) = m(\angle FBC) + m(\angle CBD) = (180 - \beta) + \beta = 180,$$

which is a contradiction. Hence $\alpha + \beta = 180$.

Theorem If, in a protractor geometry,

$$m(\angle ABC) + m(\angle CBD) = m(\angle ABD),$$

then $C \in \text{int}(\angle ABD)$.

Proof Since

$$m(\angle ABC) < m(\angle ABD),$$

it follows from an earlier result that we need only show that C and D are on the same side of $\overleftrightarrow{AB}$. Suppose C and D lie on opposite sides of $\overleftrightarrow{AB}$. Now if A and D are on the same side of $\overleftrightarrow{BC}$, then $A \in \text{int}(\angle CBD)$. Hence

$$m(\angle CBD) = m(\angle ABC) + m(\angle ABD).$$

But $m(\angle ABD) > m(\angle(CBD)$, so this is a contradiction.

If A and D are on the opposite sides of $\overleftrightarrow{BC}$, let E be a point such that $E - B - A$. Then E and D are on the same side of $\overleftrightarrow{BC}$, and so

$$m(\angle CBE) + m(\angle EBD) = m(\angle CBD).$$

Now $\angle ABC$ and $\angle CBE$ are a linear pair, so

$$m(\angle CBE) = 180 - m(\angle ABC).$$

Hence

$$180 - m(\angle ABC) + m(\angle EBD) = m(\angle CBD),$$

and so

$$180 + m(\angle EBD) = m(\angle ABC) + m(\angle CBD) = m(\angle ABD).$$

But this would mean $m(\angle ABD) > 180$, a contradiction. Hence we must have C and D on the same side of $\overleftrightarrow{AB}$.

Theorem If, in a protractor geometry, A and D lie on opposite sides of $\overleftrightarrow{BC}$ and $m(\angle ABC) + m(\angle CBD) = 180$, then $A - B - D$, that is, $\angle ABC$ and $\angle CBD$ form a linear pair.

Proof See homework.

17.2 Perpendicular lines

Definition We say lines ℓ and m in a protractor geometry are *perpendicular*, denoted $\ell \perp m$, if $\ell \cup m$ contains a right angle. We say two rays or segments are *perpendicular* if the lines they determine are perpendicular.

Theorem Given a line ℓ and a point $P \in \ell$ in a protractor geometry, there exists a unique line m through P which is perpendicular to ℓ.

Proof See homework.

Theorem Given a segment $\overline{AB}$ in a protractor geometry, there exists a unique line ℓ such that $\ell \perp \overline{AB}$ and $\ell \cap \overline{AB} = \{M\}$ where $AM = MB$.

Proof See homework.

We call the point M in the previous theorem the *midpoint* of $\overline{AB}$ and we call ℓ the *perpendicular bisector* of $\overline{AB}$.

Theorem Given an angle $\angle ABC$ in a protractor geometry, there exists a unique ray $\overrightarrow{BD}$ with $D \in \text{int}(\angle ABC)$ and $m(\angle ABD) = m(\angle DBC)$.

Proof See homework.

We call $\overrightarrow{BD}$ in the previous theorem the *angle bisector* of $\angle ABC$.

17.3 Angle congruence

Definition In a protractor geometry, we say angles $\angle ABC$ and $\angle DEF$ are *congruent*, denoted

$$\angle ABC \simeq \angle DEF,$$

if $m(\angle ABC) = m(\angle DEF)$.

Vertical Angle Theorem If, in a protractor geometry, $\angle ABC$ and $\angle DBF$ form a vertical pair, then $\angle ABC \simeq \angle DBF$.

Proof See homework

Angle Construction Theorem Given $\angle ABC$ and a ray $\overrightarrow{ED}$ which lies on the edge of a half plane H in a protractor geometry, then there exists a unique ray $\overrightarrow{EF}$ with $F \in H$ such that $\angle DEF \simeq \angle ABC$.

Proof See homework.

Angle Addition Theorem If, in a protractor geometry, $D \in \text{int}(\angle ABC)$, $S \in \text{int}(PQR)$, $\angle ABD \simeq \angle PQS$, and $\angle DBC \simeq \angle SQR$, then $\angle ABC \simeq \angle PQR$.

Proof See homework.

Angle Subtraction Theorem If, in a protractor geometry, $D \in \text{int}(\angle ABC)$, $S \in \text{int}(PQR)$, $\angle ABD \simeq \angle PQS$, and $\angle ABC \simeq \angle PQR$, then $\angle DBC \simeq \angle SQR$.