Lecture 16: The Moulton Plane

16.1 The Moulton plane

Example Given $b, m \in \mathbb{R}$, $m > 0$, let

$$M_{m,b} = \{(x, y) : (x, y) \in \mathbb{R}^2, y = mx + b \text{ for } x \leq 0, y = \frac{m}{2}x + b \text{ for } x > 0\}.$$

Let

$$\mathcal{L}_M = \{L_a : L_a \in \mathcal{L}_E, a \in \mathbb{R}\} \cup \{L_{m,b} : L_{m,b} \in \mathcal{L}_E, m \leq 0, b \in \mathbb{R}\} \cup \{M_{m,b} : m > 0, b \in \mathbb{R}\}.$$

Given distinct points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in $\mathbb{R}^2$ with $x_1 \leq x_2$, then

$$\overleftrightarrow{AB} = L_a$$

if $x_1 = x_2 = a$ and

$$\overleftrightarrow{AB} = L_{m,b}$$

if $x_1 < x_2$, $y_2 \leq y_1$,

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and

$$b = y_1 - mx_1.$$

Now suppose $y_2 > y_1$. If $x_1 < x_2 \leq 0$, then

$$\overleftrightarrow{AB} = M_{m,b},$$

and if $0 \leq x_1 < x_2$,

$$\overleftrightarrow{AB} = M_{2m,b},$$

where m and b are as above. Now if $x_1 < 0 < x_2$, we need to find m and b so that

$$y_1 = mx_1 + b$$

and

$$y_2 = \frac{m}{2}x_2 + b.$$

Subtracting and solving for m, we have

$$m = \frac{y_2 - y_1}{\frac{1}{2}x_2 - x_1}.$$

Hence, with $b = y_1 - mx_1$,

$$\overleftrightarrow{AB} = M_{m,b}.$$

To define distance, we first let

$$d_M(P,Q) = d_E(P,Q)$$

if either $P = Q$ or $\overleftrightarrow{PQ} \in \mathcal{L}_E$. If $P, Q \in M_{m,b}$ for some $m > 0$ and $b \in \mathbb{R}$, $P = (x_1, y_1)$, $Q = (x_2, y_2)$, we let

$$d_M(P,Q) = \begin{cases} d_E(P,Q), & \text{if } x_1x_2 \geq 0, \\ d_E(P,(0,b)) + d_E((0,b),Q), & \text{otherwise.} \end{cases}$$

Given a line in $\ell \in \mathcal{L}_M$, we define a ruler $f : \ell \to \mathbb{R}$ by

$$f(x,y) = y$$

if $\ell = L_a \in \mathcal{L}_E$ for some $a \in \mathbb{R}$;

$$f(x,y) = x\sqrt{1+m^2}$$

if $\ell = L_{m,b} \in L_E$ for some $m, b \in \mathbb{R}$, $m \leq 0$; and

$$f(x,y) = \begin{cases} x\sqrt{1+m^2}, & \text{if } x \leq 0, \\ x\sqrt{1+\dfrac{m^2}{4}}, & \text{if } x > 0 \end{cases}$$

if $\ell = M_{m,b}$ for some $m, b \in \mathbb{R}$, $m > 0$.

Now suppose $A, B, C \in \mathbb{R}^2$ are noncollinear. If $B \notin L_0$, we define

$$m_M(\angle ABC) = m_E(\angle A'BC'),$$

where $A' \in \overrightarrow{BA}$ and $C' \in \overrightarrow{BC}$ such that A, B, and C all lie on the same side of L_0. If $B = (0,b)$ and $P = (x,y)$, let

$$P_b = \begin{cases} (x, 2y-b), & \text{if } x > 0 \text{ and } y > b, \\ (x,y), & \text{otherwise.} \end{cases}$$

Then we define

$$m_M(\angle ABC) = m_E(\angle A_bBC_b).$$

Note that

$$2y - b = b + 2\left(\frac{y-b}{x}\right)x,$$

putting P_b where P would have been if the line $\overleftrightarrow{BP}$ had not been bent at B.

With all these definitions, $\{\mathbb{R}^2, \mathcal{L}_M, d_M, m_M\}$ is a protractor geometry, which we call the *Moulton Plane*.

Example Let $A = (-2, 1)$, $B = (0, 0)$, and $C = (1, 2)$ be points in the Moulton Plane. Then

$$\overleftrightarrow{AB} = L_{-\frac{1}{2},0},$$

$$\overleftrightarrow{BC} = M_{4,0},$$

$$\overleftrightarrow{AC} = M_{\frac{2}{5},\frac{9}{5}},$$

$$AB = \sqrt{5},$$

$$BC = \sqrt{5},$$

$$AC = d_E\left((-2,1), \left(0, \frac{9}{5}\right)\right) + d_E\left(\left(0, \frac{9}{5}\right), (1,2)\right) = \frac{2\sqrt{29}}{5} + \frac{\sqrt{26}}{5} \approx 3.1739,$$

$$m_M(\angle CAB) = \cos^{-1}\left(\frac{\langle\left(2, \frac{4}{5}\right), (2, -1)\rangle}{\frac{2\sqrt{29}}{5}\sqrt{5}}\right) = \cos^{-1}\left(\frac{8}{\sqrt{145}}\right) \approx 48.37,$$

$$m_M(\angle ABC) = \cos^{-1}\left(\frac{\langle(-2,1), (1,4)\rangle}{\sqrt{5}\sqrt{17}}\right) = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right) \approx 77.47,$$

and

$$m_M(\angle ACB) = \cos^{-1}\left(\frac{\langle(-1,-2), \left(-1, -\frac{1}{5}\right)\rangle}{\sqrt{5}\frac{\sqrt{26}}{5}}\right) = \cos^{-1}\left(\frac{7}{\sqrt{130}}\right) \approx 52.13.$$

Note that

$$m_M(\angle CAB) + m_M(\angle ABC) + m_M(\angle ACB) \approx 177.96.$$