Lecture 15: Angle Measure

15.1 Angle measure

Definition Suppose $\{\mathcal{P}, \mathcal{L}, d\}$ is a Pasch geometry and $\mathcal{A}$ is the set of all angles in this geometry. An *angle measure*, or *protractor*, is a function $m : \mathcal{A} \to (0, 180)$ such that (1) if $\overrightarrow{BC}$ lies on the edge of a half plane H and $\theta \in (0, 180)$, then there exists a unique $\overrightarrow{BA}$ with $A \in H$ and $m(\angle ABC) = \theta$, and (2) if $D \in \text{int}(\angle ABC)$, then

$$m(\angle ABC) = m(\angle ABD) + m(\angle DBC).$$

Definition Given a Pasch geometry $\{\mathcal{P}, \mathcal{L}, d\}$ and an angle measure m, we call

$$\{\mathcal{P}, \mathcal{L}, d, m\}$$

a *protractor geometry.*

Note: Given a Pasch geometry $\{\mathcal{P}, \mathcal{L}, d\}$, one may construct an angle measure m so that $\{\mathcal{P}, \mathcal{L}, d, m\}$ is a protractor geometry. We take the existence of an angle measure as an axiom for the purposes of simplifying the presentation, not because the axiom is independent of the previous axioms.

Example Given noncollinear points A, B, and C in the Euclidean Plane, let

$$m_E(\angle ABC) = \cos^{-1}\left(\frac{\langle A - B, C - B\rangle}{\|A - B\|\|C - B\|}\right).$$

We call m_E the *Euclidean angle measure.* For example, if $A = (3, 2)$, $B = (1, 0)$, and $C = (4, 0)$, then

$$\frac{\langle A - B, C - B\rangle}{\|A - B\|\|C - B\|} = \frac{\langle (2, 2), (3, 0)\rangle}{\|(2, 2)\|\|(3, 0)\|} = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}},$$

and so

$$m_E(\angle ABC) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45.$$

Example Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$ be distinct points in the Poincaré Plane. We want to define T_{BA} so that T_{BA} is a vector in the Cartesian plane tangent to the line $\overleftrightarrow{AB}$ pointing in the direction from B to A. If $x_A = x_B$, we define

$$T_{BA} = (0, y_A - y_B).$$

If $x_A \neq x_B$, then $\overleftrightarrow{AB} = {}_cL_r$ for some $c, r \in \mathbb{R}$, $r > 0$. Now if

$$(x-c)^2 + y^2 = r^2,$$

then

$$2(x-c) + 2y\frac{dy}{dx} = 0,$$

so

$$\frac{dy}{dx} = \frac{c-x}{y}.$$

Hence the slope of the tangent line at B is

$$\frac{c-x_B}{y_B}.$$

Thus we may take

$$T_{BA} = \begin{cases} (y_B, c-x_B), & \text{if } x_B < x_A, \\ -(y_B, c-x_B), & \text{if } x_B > x_A. \end{cases}$$

Given noncollinear points A, B, and C in the Poincaré Plane, let

$$m_H(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\|T_{BC}\|}\right).$$

We call m_H the *Poincaré angle measure*. For example, if $A = (1, 10)$, $B = (1, 5)$, and $C = (4, 4)$, then $A, B \in {}_1L$, so

$$T_{BA} = (0, 5),$$

and $B, C \in {}_1L_5$, so

$$T_{BC} = (5, 0).$$

Thus

$$\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\|T_{BC}\|} = \frac{\langle (0,5), (5,0)\rangle}{\|(0,5)\|\|(5,0)\|} = 0,$$

and so

$$m_H(\angle ABC) = \cos^{-1}(0) = 90.$$