# Lecture 14: Convex Quadrilaterals

## 14.1 Convex quadrilaterals

**Definition** Suppose $\{A, B, C, D\}$ are four distinct points in a metric geometry, no three of which are collinear, and $\text{int}(\overline{AB})$, $\text{int}(\overline{BC})$, $\text{int}(\overline{CD})$, and $\text{int}(\overline{DA})$ are disjoint. We call

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$$

a *quadrilateral.*

**Theorem** Given a quadrilateral $\square ABCD$ in a metric geometry, then

$$\begin{aligned} \square ABCD &= \square BCDA = \square CDAB = \square DABC \\ &= \square ADCB = \square DCBA = \square CBAD = \square BADC. \end{aligned}$$

If both $\square ABCD$ and $\square ABDC$ are quadrilaterals, then $\square ABCD \neq \square ABDC$.

**Proof** See homework.

**Theorem** If, in a metric geometry, $\square ABCD = \square PQRS$, then

$$\{A, B, C, D\} = \{P, Q, R, S\}.$$

Moreover, if $A = P$, then $C = R$ and either $B = Q$ or $B = S$.

**Definition** Given a quadrilateral $\square ABCD$, we call $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ the *sides* of $\square ABCD$; $A$, $B$, $C$, and $D$ the *vertices* of $ABCD$; $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$ the *angles* of $\square ABCD$; and $\overline{AC}$ and $\overline{BD}$ the *diagonals* of $\square ABCD$. We call the endpoints of a diagonal *opposite vertices*, two sides which have a common vertex *adjacent sides*, two sides which do not have a common vertex *opposite sides*, two angles which contain a common side *adjacent angles*, and two angles which do not contain a common side *opposite sides*.

**Definition** We call $\square ABCD$ in a Pasch geometry a *convex quadrilateral* if each side of $\square ABCD$ lies on a half plane determined by its opposite side.

**Theorem** In a Pasch geometry, a quadrilateral is a convex quadrilateral if and only if the vertex of each angle is in the interior of its opposite angle.

**Proof** See homework.

**Theorem** If, in a Pasch geometry, $\square ABCD$ is a convex quadrilateral, then $\overline{AC} \cap \overline{BD} \neq \emptyset$.

**Proof** Since, by the previous theorem, $D \in \text{int}(\angle ABC)$, $\overrightarrow{BD} \cap \overline{AC} = \{E\}$ for some point $E$ with $A - E - C$ by the Crossbar Theorem. Also, $C \in \text{int}(\angle DAB)$, so, again by the Crossbar Theorem, $\overrightarrow{AC} \cap \overline{BD} = \{F\}$ for some point $F$ with $B - F - D$. Since $\overleftrightarrow{AC}$ intersects $\overleftrightarrow{BD}$ in at most one point, $E = F$, and we have $\overline{AC} \cap \overline{BD} = \{E\}$.

**Theorem** If, in a Pasch geometry, $\square ABCD$ is a quadrilateral and $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{AD}$, then $\square ABCD$ is a convex quadrilateral.

**Proof** $\overleftrightarrow{BC} \cap \overleftrightarrow{AD} = \emptyset$, so $\overline{BC}$ lies on one side of $\overleftrightarrow{AD}$ and $\overline{AD}$ lies on one side of $\overleftrightarrow{BC}$. Suppose $\overline{AB}$ does not lie on one side of $\overleftrightarrow{CD}$. Then $\overline{AB} \cap \overleftrightarrow{CD} = \{E\}$ for some point $E$ with $A - E - B$. Now if $H$ is the side of $\overleftrightarrow{BC}$ which contains $\overline{AD}$ and $G$ is the side of $\overleftrightarrow{AD}$ which contains $\overline{BC}$, then $E \in H$ and $E \in G$. Since $E \in H$, we cannot have $D - C - E$; since $E \in G$, we cannot have $C - D - E$. Thus $C - E - D$. But then $\text{int}(\overline{AB}) \cap \text{int}(\overline{CD}) \neq \emptyset$, contradicting the assumption that $\square ABCD$ is a quadrilateral. Hence $\overline{AB}$ lies on one side of $\overleftrightarrow{CD}$. A similar argument shows that $\overline{CD}$ lies on one side of $\overleftrightarrow{AB}$. Hence $\square ABCD$ is a convex quadrilateral.