Lecture 13: Interiors

13.1 Interiors

Definition If A and B are distinct points in a metric geometry, we call

$$\text{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\}$$

the *interior* of the ray $\overrightarrow{AB}$ and we call

$$\text{int}(\overline{AB}) = \overline{AB} - \{A, B\}$$

the *interior* of the segment $\overline{AB}$.

Theorem If A and B are distinct points in a metric geometry, then $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, $\overline{AB}$, $\text{int}(\overrightarrow{AB})$, and $\text{int}(\overline{AB})$ are all convex sets.

Proof Various homework exercises.

Theorem If, in a Pasch geometry, ℓ is a line and $\mathcal{S}$ is a nonempty convex set of points with $\mathcal{S} \cap \ell = \emptyset$, then all the points in $\mathcal{S}$ lie on the same side of ℓ.

Proof Immediate from the definition of convex set.

Theorem Let $\mathcal{S}$ be a line, a ray, a segment, the interior of a ray, or the interior of a segment in a Pasch geometry. If ℓ is a line with $\mathcal{S} \cap \ell = \emptyset$, then all of $\mathcal{S}$ lies on one side of ℓ. If A, B, and C are points with $A - B - C$ and $\overleftrightarrow{AB} \cap \ell = \{B\}$, then $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overline{AB})$ both lie on the same side of ℓ, while $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overrightarrow{BC})$ lie on opposite sides of ℓ.

Proof Consequence of previous theorem and definition.

Z Theorem If, in a Pasch geometry, P and Q lie on opposite sides of a line $\overleftrightarrow{AB}$, then $\overrightarrow{BP} \cap \overrightarrow{AQ} = \emptyset$.

Proof By the previous theorem, $\text{int}(\overrightarrow{BP})$ and $\text{int}(\overrightarrow{AQ})$ lie on opposite sides of $\overleftrightarrow{AB}$. Thus $\text{int}(\overrightarrow{BP}) \cap \text{int}(\overrightarrow{AQ}) = \emptyset$. Now A, B, and Q are noncollinear, so $B \notin \overleftrightarrow{AQ}$, and thus $\overrightarrow{BP} \cap \text{int}(\overrightarrow{AQ}) = \emptyset$. Similarly, $A \notin \overleftrightarrow{BP}$, so $\overrightarrow{BP} \cap \overrightarrow{AQ} = \emptyset$.

Definition Given noncollinear points A, B, and C in a Pasch geometry, let H be the side of $\overleftrightarrow{AB}$ which contains C and let G be the side of $\overleftrightarrow{BC}$ which contains A. We call

$$\text{int}(\angle ABC) = H \cap G$$

the *interior* of $\angle ABC$.

Theorem If, in a Pasch geometry, $\angle ABC = \angle DEF$, then $\text{int}(\angle ABC) = \text{int}(\angle DEF)$.

Proof We know that $B = E$ and either $\overrightarrow{BA} = \overrightarrow{ED}$ or $\overrightarrow{BA} = \overrightarrow{EF}$. Suppose $\overrightarrow{BA} = \overrightarrow{ED}$. Let H be the side of $\overleftrightarrow{AB}$ which contains C and let G be the side of $\overleftrightarrow{BC}$ which contains A. Now $D \in \overrightarrow{BA}$, so A and D are on same side of $\overleftrightarrow{BC} = \overleftrightarrow{EF}$. Hence $D \in G$. $F \in \overrightarrow{BC}$, so C and F are on the same side of $\overleftrightarrow{AB} = \overleftrightarrow{DE}$. Hence $F \in H$. Thus

$$\text{int}(\angle ABC) = H \cap G = \text{int}(\angle DEF).$$

Theorem In a Pasch geometry, $P \in \text{int}(\angle ABC)$ if and only if A and P are on the same side of $\overleftrightarrow{BC}$ and C and P are on the same side of $\overleftrightarrow{AB}$.

Proof See homework.

Theorem In a Pasch geometry, $\text{int}(\overline{AC}) \subset \text{int}(\angle ABC)$.

Proof See homework.

13.2 Crossbar

Crossbar Theorem If, in a Pasch geometry, $P \in \text{int}(\angle ABC)$, then $\overrightarrow{BP} \cap \overline{AC} = \{F\}$ where $A - F - C$.

Proof Let E be a point such that $E - B - C$. We first show that $\overleftrightarrow{BP} \cap \overline{AE} = \emptyset$. Now P and C are on the same side of $\overleftrightarrow{AB}$ and C and E are on opposite sides of $\overleftrightarrow{AB}$, so P and E are on opposite sides of $\overleftrightarrow{AB}$. Hence, by the Z Theorem, $\overrightarrow{BP} \cap \overline{AE} = \emptyset$. Now let Q be a point such that $Q - B - P$. Then Q and P are on opposite sides of $\overleftrightarrow{BC}$ and P and A are on the same side of $\overleftrightarrow{BC}$, so Q and A are on opposite sides of $\overleftrightarrow{BC} = \overleftrightarrow{EC}$. Hence, by the Z Theorem, $\overrightarrow{BQ} \cap \overline{AE} = \emptyset$. Thus $\overleftrightarrow{BP} \cap \overline{AE} = \emptyset$.

Applying Pasch's Postulate to $\triangle ECA$, we conclude that $\overleftrightarrow{BP} \cap \overline{AC} \neq \emptyset$. Since A, B, and C are noncollinear, we must have $\overleftrightarrow{BP} \cap \overline{AC} = \{F\}$ for some F. Now $F \neq A$ (since $\overleftrightarrow{BP} \cap \overline{AE} = \emptyset$) and $F \neq C$ (since $P \notin \overleftrightarrow{BC}$). Thus $A - F - C$. Finally, P and A are on the same side of $\overleftrightarrow{BC}$ and A and F are on the same side of $\overleftrightarrow{BC}$, so P and F are on the same side of $\overleftrightarrow{BC}$. Hence $F \in \overrightarrow{BP}$.

Theorem If, in a Pasch geometry, $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$, then $P \in \text{int}(\angle ABC)$ if and only if A and C are on opposite sides of $\overleftrightarrow{BP}$.

Proof See homework.

Theorem If, in a Pasch geometry, $A - B - D$, then $P \in \text{int}(\angle ABC)$ if and only if $C \in \text{int}(\angle DBP)$.

Proof See homework.

Definition If A, B and C are noncollinear points in a Pasch geometry and H is the side of $\overleftrightarrow{AB}$ which contains C, G is the side of $\overleftrightarrow{BC}$ which contains A, and I is the side of $\overleftrightarrow{AC}$ which contains B, then we call

$$\text{int}(\triangle ABC) = G \cap H \cap I$$

the *interior* of $\triangle ABC$.

Theorem In a Pasch geometry, $\text{int}(\triangle ABC)$ is convex.

Proof See homework.