Lecture 12: Pasch's Postulate

12.1 Pasch's Postulate

Pasch's Theorem Suppose $\{\mathcal{P}, \mathcal{L}, d\}$ is a metric geometry which satisfies the plane separation axiom. If ℓ is a line, $\triangle ABC$ is a triangle, $D \in \ell$, and $A - D - B$, then either $\ell \cap \overline{AC} \neq \emptyset$ or $\ell \cap \overline{BC} \neq \emptyset$.

In other words, in a metric geometry satisfying PSA, a line intersecting a triangle intersects at least two sides of the triangle.

Proof Suppose $A - D - B$, $D \in \ell \cap \triangle ABC$, and $\ell \cap \overline{AC} = \emptyset$. First note that $A \in \overline{AC}$, so $A \notin \ell$. Thus $\ell \neq \overleftrightarrow{AB}$, and so $B \notin \ell$. Hence A and B are on opposite sides of ℓ. Since A and C are on the same side of ℓ (since $\ell \cap \overline{AC} = \emptyset$), it follows that C and B are on opposite sides of ℓ. Hence $\overline{BC} \cap \ell \neq \emptyset$.

Definition We say a metric geometry satisfies *Pasch's Postulate* (PP) if whenever ℓ is a line, $\triangle ABC$ is a triangle, $D \in \ell$, and $A - D - B$, then either $\ell \cap \overline{AC} \neq \emptyset$ or $\ell \cap \overline{BC} \neq \emptyset$.

Note that the above theorem says that in a metric geometry, PSA implies PP.

Theorem Suppose $\{\mathcal{P}, \mathcal{L}, d\}$ is a metric geometry satisfying Pasch's Postulate. If A, B, and C are noncollinear points and $\ell \cap \{A, B, C\} = \emptyset$, then ℓ cannot intersect all three sides of $\triangle ABC$.

Proof Suppose $\{D\} = \ell \cap \overline{AB}$, $\{E\} = \ell \cap \overline{AC}$, and $\{F\} = \ell \cap \overline{BC}$. Suppose $D - E - F$. Now $\overleftrightarrow{BD} = \overleftrightarrow{AB}$ and $\overleftrightarrow{BF} = \overleftrightarrow{BC}$, so B, D, and F are not collinear. Now $\overleftrightarrow{AC} \cap \overline{DF} = \{E\}$, so, by Pasch's Postulate, we must have either $\overleftrightarrow{AC} \cap \overline{BD} \neq \emptyset$ or $\overleftrightarrow{AC} \cap \overline{BF} \neq \emptyset$. But

$$\overleftrightarrow{AC} \cap \overline{BD} \subset \overleftrightarrow{AC} \cap \overleftrightarrow{BD} = \{A\}$$

and $A \neq \overline{BD}$ (since $A - D - B$), so $\overleftrightarrow{AC} \cap \overline{BD} = \emptyset$. Also,

$$\overleftrightarrow{AC} \cap \overline{BF} \subset \overleftrightarrow{AC} \cap \overleftrightarrow{BF} = \{C\}$$

and $C \neq \overline{BF}$ (since $B - F - C$), so $\overleftrightarrow{AC} \cap \overline{BF} = \emptyset$. Hence our assumptions imply a contradiction of Pasch's Postulate.

Theorem A metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$ satisfying Pasch's Postulate also satisfies the plane separation axiom.

Proof Given a line ℓ, let P be a point not on ℓ and define

$$H_1 = \{Q : Q \in \mathcal{P}, Q = P \text{ or } \overline{PQ} \cap \ell = \emptyset\}$$

and

$$H_2 = \{Q : Q \in \mathcal{P}, Q \notin \ell, \overline{PQ} \cap \ell \neq \emptyset\}.$$

Clearly, H_1 and H_2 are disjoint and $\mathcal{P} - \ell = H_1 \cup H_2$.

We first show H_1 is convex. Let $R, S \in H_1$ and suppose $R - T - S$. If P, R, and S are collinear, then

$$\overline{RS} \subset \overline{PR} \cup \overline{PS} \subset H_1,$$

so $T \in H_1$. If P, R, and S are noncollinear, then $\ell \cap \overline{PR} = \emptyset$ and $\ell \cap \overline{PS} = \emptyset$, with Pasch's Postulate applied to $\triangle PRS$, imply that $\ell \cap \overline{RS} = \emptyset$. But then $\ell \cap \overline{PR} = \emptyset$ and $\ell \cap \overline{RT} = \emptyset$, with Pasch's Postulate applied to $\triangle PRT$, imply that $\ell \cap \overline{PT} = \emptyset$. Thus $T \in H_1$. Hence H_1 is convex.

We next show that H_2 is convex. Let $R, S \in H_2$ and suppose $R - T - S$. If P, R, and S are collinear, then either

$$\overline{PR} \subset \overline{PT}$$

or

$$\overline{PS} \subset \overline{PT}.$$

Since both $\overline{PR} \cap \ell \neq \emptyset$ and $\overline{PS} \cap \ell \neq \emptyset$, it follows that $\overline{PT} \cap \ell \neq \emptyset$, so, since $T \notin \ell$, $T \in H_2$. If P, R, and S are noncollinear, first note that $T \notin \ell$, since that would imply that ℓ intersects all three sides of $\triangle PRS$. Indeed, we must have $\ell \cap \overline{RS} = \emptyset$. In particular, $\ell \cap \overline{RT} = \emptyset$; combined with $\ell \cap \overline{PR} \neq \emptyset$ and Pasch's Postulate applied to $\triangle PRT$, this implies $\ell \cap \overline{PT} \neq \emptyset$. Thus $T \in H_2$. Hence H_2 is convex.

Now suppose $R \in H_1$ and $S \in H_2$. We need to show that $\overline{RS} \cap \ell \neq \emptyset$. If $R = P$, then

$$\overline{RS} \cap \ell = \overline{PS} \cap \ell \neq \emptyset,$$

and we are done. So assume $R \neq P$. If R, S and P are collinear, then either $P - R - S$, $R - P - S$, or $P - S - R$. In the first case, $\overline{PS} \cap \ell \neq \emptyset$ and $\overline{PR} \cap \ell = \emptyset$ imply $\overline{RS} \cap \ell \neq \emptyset$; in the second case, $\overline{PS} \cap \ell \neq \emptyset$ and $\overline{PS} \subset \overline{RS}$ imply $\overline{RS} \cap \ell \neq \emptyset$; and the third case cannot occur because $\overline{PS} \subset \overline{PR}$ would imply $\overline{PR} \cap \ell \neq \emptyset$. If R, S, and P are noncollinear, then $\overline{PS} \cap \ell \neq \emptyset$ and $\overline{PR} \cap \ell = \emptyset$ imply, with Pasch's Postulate applied to $\triangle PRS$, that $\overline{RS} \cap \ell \neq \emptyset$.

12.2 Pasch geometries

Definition We call a metric geometry satisfying Pasch's Postulate (or, equivalently, satisfying the plane separation axiom) a *Pasch Geometry.*

Example The Euclidean Plane, the Poincaré Plane, and the Taxicab Plane are all Pasch Geometries.

Example Define a metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$ as follows: We let

$$\mathcal{P} = \{(x, y) : (x, y) \in \mathbb{R}^2, x < 0 \text{ or } x \geq 1\},$$

and

$$\mathcal{L} = \{\ell \cap \mathcal{P} : \ell \in \mathcal{L}_E, \ell \cap \mathcal{P} \neq \emptyset\}.$$

To define d, we first define rulers. If $\ell = L_a$, $a \in \mathbb{R}$ with $a < 0$ or $a \geq 1$, define

$$f(x, y) = y$$

for all $(x, y) \in \ell$. If $\ell = L_{m,b}$, $m \in \mathbb{R}$, $b \in \mathbb{R}$, define

$$f(x, y) = \begin{cases} x\sqrt{1 + m^2}, & \text{if } x < 0, \\ (x - 1)\sqrt{1 + m^2}, & \text{if } x \geq 1. \end{cases}$$

Given any two distinct points P and Q in $\mathcal{P}$, let f be the ruler as described above for $\overleftrightarrow{PQ}$ and let

$$d(P, Q) = |f(P) - f(Q)|.$$

With these definitions, $\{\mathcal{P}, \mathcal{L}, d\}$ is a metric geometry.

Now consider $\triangle ABC$ where $A = (-1, 0)$, $B = (2, 0)$, and $C = (2, 3)$. If ℓ is the line with equation $y = 1.5$ (that is, $\ell = L_{0,1.5}$), then $\ell \cap \overline{BC} = \{(2, 1.5)\}$. However, $\ell \cap \overline{AC} = \emptyset$ and $\ell \cap \overline{AB} = \emptyset$. Thus $\{\mathcal{P}, \mathcal{L}, d\}$ is not a Pasch Geometry. We call $\{\mathcal{P}, \mathcal{L}, d\}$ the *Missing Strip Plane*.