Lecture 11: PSA for the Euclidean and Hyperbolic Planes

11.1 PSA for the Euclidean Plane

Notation: If $P = (x, y) \in \mathbb{R}^2$, we let $P^\perp = (-y, x)$. Note then that, for any $P \in \mathbb{R}^2$,

$$\langle P, P^\perp \rangle = 0.$$

Moreover, suppose $P = (x, y) \neq (0, 0)$, $Q = (u, v)$, and

$$\langle Q, P^\perp \rangle = 0.$$

Then

$$-uy + vx = 0.$$

If $x \neq 0$, then

$$v = \frac{uy}{x},$$

and so

$$Q = (u, v) = u\left(1, \frac{y}{x}\right) = \frac{u}{x}(x, y) = tP$$

where $t = \frac{u}{x}$. Similarly, if $y \neq 0$,

$$u = \frac{vx}{y},$$

and

$$Q = (u, v) = v\left(\frac{x}{y}, 1\right) = \frac{v}{y}(x, y) = tP$$

where $t = \frac{v}{y}$. In either case, we have $\langle Q, P^\perp \rangle = 0$ implies Q is a scalar multiple of P.

Theorem If P and Q are distinct points in the Euclidean Plane, then

$$\overleftrightarrow{PQ} = \{R : R \in \mathbb{R}^2, \langle R - P, (Q - P)^\perp \rangle = 0\}.$$

Proof Let $A \in \overleftrightarrow{PQ}$. Then

$$A = P + t(Q - P)$$

for some $t \in \mathbb{R}$. Hence

$$\langle A - P, (Q - P)^\perp \rangle = \langle t(Q - P), (Q - P)^\perp \rangle = t\langle (Q - P), (Q - P)^\perp \rangle = 0,$$

so

$$A \in \{R : R \in \mathbb{R}^2, \langle R - P, (Q - P)^\perp \rangle = 0\}.$$

Now suppose

$$A \in \{R : R \in \mathbb{R}^2, \langle R - P, (Q - P)^\perp \rangle = 0\}.$$

Then

$$A - P = t(Q - P)$$

for some $t \in \mathbb{R}$, that is, $A = P + t(Q - P)$. Hence $A \in \overleftrightarrow{PQ}$. Thus

$$\overleftrightarrow{PQ} = \{R : R \in \mathbb{R}^2, \langle R - P, (Q - P)^\perp \rangle = 0\}.$$

Definition If P and Q are distinct points in the Euclidean Plane and $\ell = \overleftrightarrow{PQ}$, then we call

$$H^+ = \{A : A \in \mathbb{R}^2, \langle A - P, (Q - P)^\perp \rangle > 0\}$$

and

$$H^- = \{A : A \in \mathbb{R}^2, \langle A - P, (Q - P)^\perp \rangle < 0\}$$

the *Euclidean half planes* determined by ℓ.

Theorem Given $\ell = \overleftrightarrow{PQ}$ in the Euclidean Plane, the half planes H^+ and H^- are convex.

Proof Let $A, B \in H^+$. Let $C \in \overline{AB}$. Then

$$C = A + t(B - A) = (1 - t)A + tB$$

for some $0 < t < 1$. Hence

$$\begin{aligned}\langle C - P, (Q - P)^\perp \rangle &= \langle ((1 - t)A + tB) - ((1 - t)P + tP), (Q - P)^\perp \rangle \\ &= (1 - t)\langle (A - P), (Q - P)^\perp \rangle + t\langle (B - P), (Q - P)^\perp \rangle \\ &> 0,\end{aligned}$$

so $C \in H^+$ and H^+ is convex. The proof for H^- is similar.

Theorem The Euclidean Plane $\{\mathbb{R}^2, \mathcal{L}_E, d_E\}$ satisfies the plane separation axiom.

Proof Given $\ell \in \mathcal{L}_E$, let H^+ and H^- be the Euclidean half planes determined by ℓ. Since for every $A \in \mathbb{R}^2$, exactly one of

$$\langle A - P, (Q - P)^\perp \rangle < 0,$$

$$\langle A - P, (Q - P)^\perp \rangle = 0,$$

or

$$\langle A - P, (Q - P)^\perp \rangle > 0,$$

holds, it follows that H^+, H^-, and ℓ are disjoint and

$$\mathbb{R}^2 - \ell = H^+ \cup H^-.$$

Moreover, we have already seen that H^+ and H^- are convex. It remains to show that given $A \in H^+$ and $B \in H^-$,

$$\overline{AB} \cap \ell \neq \emptyset.$$

Define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = \langle (A + t(B - A)) - P, (Q - P)^\perp \rangle.$$

Then

$$g(0) = \langle A - P, (Q - P)^\perp \rangle > 0$$

and

$$g(1) = \langle B - P, (Q - P)^\perp \rangle < 0.$$

Since g is continuous (in fact, a first degree polynomial), it follows from the Intermediate Value Theorem that there exists $s \in (0,1)$ such that $g(s) = 0$. If $C = A + s(B - A)$, then $C \in \overline{AB}$ and $C \in \ell$. Hence $\overline{AB} \cap \ell \neq \emptyset$.

Theorem The Taxicab Plane $\{\mathbb{R}^2, \mathcal{L}_E, d_T\}$ satisfies the Plane Separation Axiom.

Proof Follows from the fact that the Taxicab Plane and the Euclidean Plane have the same lines and the same betweenness relation.

11.2 PSA for the Poincaré Plane

Definition Given a line $\ell = {}_aL$ in the Poincaré Plane, we call

$$H^+ = \{(x,y) : (x,y) \in \mathbb{H}, x > a\}$$

and

$$H^- = \{(x,y) : (x,y) \in \mathbb{H}, x < a\}$$

the *Poincaré half planes* determined by ℓ. Given a line $\ell = {}_cL_r$ in the Poincaré Plane, we call

$$H^+ = \{(x,y) : (x,y) \in \mathbb{H}, (x-c)^2 + y^2 > r^2\}$$

and

$$H^- = \{(x,y) : (x,y) \in \mathbb{H}, (x-c)^2 + y^2 < r^2\}$$

the *Poincaré half planes* determined by ℓ.

Theorem Given a line ℓ in the Poincaré Plane, H^+ and H^- are convex sets.

Proof Suppose $\ell = {}_aL$ and $A = (x_1, y_1) \in H^+$, $B = (x_2, y_2) \in H^+$ are distinct points. If $P = (x, y) \in \overline{AB}$, then either $x_1 \le x \le x_2$ or $x_2 \le x \le x_1$. Thus, since $x_1 > a$ and $x_2 > a$, $x > a$, so $P \in H^+$ and H^+ is convex. Similarly, H^- is convex.

Now suppose $\ell = {}_cL_r$ and $A = (x_1, y_1) \in H^+$, $B = (x_2, y_2) \in H^+$ are distinct points. Now if $\overleftrightarrow{AB} = {}_aL$ for some $a \in \mathbb{R}$, then $f : \mathbb{R} \to \overleftrightarrow{AB}$ defined by

$$f(t) = (a, e^t)$$

parametrizes $\overleftrightarrow{AB}$. In particular, there exist $t_A, t_B \in \mathbb{R}$ such that $f(t_A) = A$ and $f(t_B) = B$. In this case define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = (a - c)^2 + e^{2t} - r^2.$$

Note that $g(t) < 0$ if and only if $f(t) \in H^-$, $g(t) = 0$ if and only if $f(t) \in \ell$, and $g(t) > 0$ if and only if $f(t) \in H^+$. Moreover,

$$g'(t) = 2e^{2t} > 0$$

for all t, so g is an increasing function. Now if $C \in \overline{AB}$, $A - C - B$, so there exists a t_C, between t_A and t_B, for which $f(t_C) = C$. Since $g(t_A) > 0$, $g(t_B) > 0$, and g is increasing, it follows that $g(t_C) > 0$.

Finally, suppose $\overleftrightarrow{AB} = {}_dL_s$. We may parametrize $\overleftrightarrow{AB}$ with $f : \mathbb{R} \to \overleftrightarrow{AB}$ defined by

$$f(t) = (d + s\tanh(t), s\operatorname{sech}(t)).$$

Let $t_A, t_B \in \mathbb{R}$ such that $f(t_A) = A$ and $f(t_B) = B$. Now define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = (d + s\tanh(t) - c)^2 + s^2\operatorname{sech}^2(t) - r^2.$$

As above, $g(t) < 0$ if and only if $f(t) \in H^-$, $g(t) = 0$ if and only if $f(t) \in \ell$, and $g(t) > 0$ if and only if $f(t) \in H^+$. Now

$$\begin{aligned} g'(t) &= 2(d + s\tanh(t) - c)s\operatorname{sech}^2(t) - 2s^2\operatorname{sech}(t)\operatorname{sech}(t)\tanh(t) \\ &= (2d + 2s\tanh(t) - 2c - 2s\tanh(t))s\operatorname{sech}^2(t) \\ &= 2(d - c)s\operatorname{sech}^2(t). \end{aligned}$$

Thus g is increasing if $d - c > 0$, constant if $d = c$, and decreasing if $d - c < 0$. It follows that if $C \in \overline{AB}$ and $f(t_C) = C$, then, since t_C is between t_A and t_B, $g(t_C) > 0$ and $C \in H^+$. Thus H^+ is convex. A similar argument shows that H^- is convex.

Theorem The Poincaré Plane $\{\mathbb{H}, \mathcal{L}_H, d_H\}$ satisfies the plane separation axiom.

Proof Use the functions f and g defined in the previous proof, combined with the Intermediate Value Theorem.