Lecture 10: Plane Separation

10.1 Convex sets

Definition We say a set of points S in a metric geometry is *convex* if for every two distinct points $P, Q \in S$, $\overline{PQ} \subset S$.

Example The set
$$S = \{(x, y) : (x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\}$$
is convex set in the Euclidean Plane. For if $P, Q \in S$, then $\|P\| \leq 1$ and $\|Q\| \leq 1$. If $R \in \overline{PQ}$, then
$$R = A + t(B - A)$$
for some $0 < t < 1$. Hence
$$\|R\| = \|A + t(B - A)\| \leq (1 - t)\|A\| + t\|B\| \leq \max\{\|A\|, \|B\|\} \leq 1,$$
and so $R \in S$.

Example The set
$$S = \{(x, y) : (x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$$
is not convex in the Euclidean Plane. For example, if we let $P = (1, 0)$ and $Q = (-1, 0)$, then $(0, 0) \in \overline{PQ}$, but $P \notin S$.

10.2 The plane separation axiom

Definition We say a metric geometry $\{\mathcal{P}, \mathcal{L}, d\}$ satisfies the *plane separation axiom* (PSA) if for every $\ell \in \mathcal{L}$, there exist $H_1 \subset \mathcal{P}$ and $H_2 \subset \mathcal{P}$ such that (1) $H_1 \cup H_2 = \mathcal{P} - \ell$, (2) $H_1 \cap H_2 = \emptyset$, (3) H_1 and H_2 are convex, and (4) if $P \in H_1$ and $Q \in H_2$, then $\overline{PQ} \cap \ell \neq \emptyset$. We call H_1 and H_2 the *half planes* determined by ℓ.

Theorem Let ℓ be a line in a metric geometry. If H_1, H_2 and H_1', H_2' satisfy the conditions of the plane separation axiom for the line ℓ, then either $H_1 = H_1'$ and $H_2 = H_2'$, or $H_1 = H_2'$ and $H_2 = H_1'$.

Proof At least one of H_1 and H_2 is nonempty. Suppose $P \in H_1$. Since $P \notin \ell$, either $P \in H_1'$ or $P \in H_2'$. Suppose $P \in H_1'$. We will show that $H_1 = H_1'$. Let $Q \in H_1$, $Q \neq P$. Then $Q \notin \ell$ and if $Q \in H_2'$, then $\overline{PQ} \cap \ell \neq \emptyset$, contradicting the convexity of H_1. Thus $Q \in H_1'$ and $H_1 \subset H_1'$. The argument now works in the other direction: if $Q \in H_1'$, then $Q \notin \ell$ and $Q \notin H_2$. Hence $H_1' \subset H_1$, and so $H_1 = H_1'$.

It now follows that
$$H_2 = \mathcal{P} - \ell - H_1 = \mathcal{P} - \ell - H_1' = H_2'.$$

Definition Let ℓ be a line in metric geometry which satisfies PSA and let H_1 and H_2 be the half planes determined by ℓ. We say points P and Q *lie on the same side* of ℓ if either $P, Q \in H_1$ or $P, Q \in H_2$. We say P and Q *lie on opposite sides* of ℓ if either $P \in H_1$ and $Q \in H_2$ or $P \in H_2$ and $Q \in H_1$. If $P \in H_1$, we say H_1 is the *side* of ℓ which contains P.

Theorem Given a line ℓ and two points P and Q not on ℓ in a metric geometry which satisfies PSA, then P and Q are on opposite sides of ℓ if and only if $\overline{PQ} \cap \ell \neq \emptyset$, and P and Q are on the same side of ℓ if and only if either $P = Q$ or $\overline{PQ} \cap \ell = \emptyset$.

Proof Follows directly from the definitions.

Theorem Let ℓ be a line in a metric geometry which satisfies PSA. If P and Q are on opposite sides of ℓ and Q and R are on opposite sides of ℓ, then P and R are on the same side of ℓ.

Proof Homework.

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Proof Homework.

Theorem If ℓ and ℓ' are lines in a metric geometry satisfying PSA and H is a half plane of both ℓ and ℓ', then $\ell = \ell'$.

Proof Suppose $\ell \neq \ell'$. Let $P \in \ell - \ell'$ and $Q \in \ell' - \ell$ (there exist such P and Q since $\ell \cap \ell'$ can contain at most one point). Let $A, B \in \overleftrightarrow{PQ}$ such that $A - P - Q$ and $P - Q - B$. Then $A - P - Q - B$. In particular, $A - P - B$, so A and B are on opposite sides of ℓ. Hence either $A \in H$ or $B \in H$. Suppose $A \in H$. Now $A - P - Q$ implies A and P are on the same side of ℓ'. Hence $P \in H$, contradicting $P \in \ell$. Hence we must have $\ell = \ell'$.

Definition If H is a half plane determined by a line ℓ, then we call ℓ the *edge* of H.