

Lecture 1: Axioms and Models

1.1 Geometry

Although the study of geometry dates back at least to the early Babylonian and Egyptian societies, our modern systematic approach to the subject originates in the work of Euclid (325? BC - 265? BC). The *Elements*, written by Euclid around 300 B.C., is perhaps the most influential mathematics book ever written. Here Euclid succinctly unifies much of the Greek knowledge of geometry into a single work, all deduced from ten well chosen axioms.

Note: All translations below are from Sir Thomas L. Heath, and were taken from *The Perseus Project*, <http://www.perseus.tufts.edu>, Gregory Crane, editor.

The *Elements* begins with 23 definitions:

1. A point is that which has no part.
2. A line is breadthless length.
3. The extremities of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilinear.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A boundary is that which is an extremity of anything.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all the straight lines falling

upon it from one point among those lying within the figure are equal to one another;

16. And the point is called the centre of the circle.
17. A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.
19. Rectilineal figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.
20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.
21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.
22. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.
23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

The definitions are followed by 10 axioms, sometimes divided into 5 *postulates*:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Followed by 5 *common notions*:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

1.2 Comments on definitions

Can everything be defined? That is, without going in a circle? For example, if a point is “that which has no part,” what is a part? And can you define “part” without referring to a “point”?

What does it mean to say that a line is “that which lies evenly with itself?”

Clearly, Euclid has in mind real objects. He is trying to get the reader to identify certain terms with specific objects, objects he assumes the reader is able to comprehend with the help of the descriptions in the definitions. Perhaps he is prodding our memory of things known *a priori* (corresponding to the Platonic theory of knowledge).

1.3 Comments on the postulates

Are the postulates self-evident? Are they facts that we can easily abstract from the physical world and hold with confidence about the world of mathematical objects?

Is the fifth postulate as self-evident as the other four?

It may be shown that the statement

Given a point P not on a line ℓ , there exists a unique line through P parallel to ℓ .

is equivalent to Euclid’s fifth postulate. In part, Euclid is trying to avoid references to infinity in stating the postulate this way. In particular, what he calls straight lines are what we would call line segments.

For the next 2000 years, mathematicians try to show that the fifth postulate is not needed, that in fact it can be derived from the other axioms.

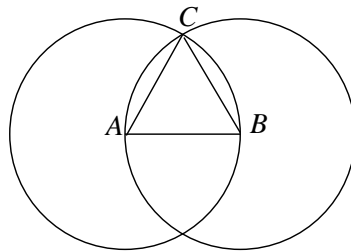
1.4 Comments on the common notions

The common notions are axioms which are thought to hold for all areas of reasoning, not just geometry.

How self-evident is the fifth common notion? Does this axiom hold for infinite sets? This in fact may be another attempt by Euclid to exclude the infinite from his considerations.

1.5 Some example propositions

Proposition 1 On a given finite straight line to construct an equilateral triangle.



Proof Let A and B be the endpoints of the given straight line. Construct a circle of radius AB with center at A and a circle of radius BA with center at B . Let C be a point where the two circles intersect. Then $\triangle ABC$ is an equilateral triangle, since, by construction, $AB = AC = CB$.

Question: Which axiom guarantees that the two circles intersect?

Proposition 4 If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

This is the famous Side-Angle-Side, or SAS, theorem. Euclid's proof is controversial, and this statement is now typically taken as an axiom.

1.6 Axioms and models

In the modern view of geometry, the fundamental objects, that is, points and lines, are undefined, the only requirement being that they satisfy certain axioms. In other words, a *geometry*, of which there are many, will consist of two sets, a set of points and set of lines, which satisfy a given set of axioms. We do not think of the axioms as self-evident truths, but rather as basic assumptions that we make in order to get our study started. Indeed, different axioms are useful in different circumstances. What we do require is that the axioms be *consistent*, that is, they must not imply a contradiction, and that they be fruitful.

A *model* for a geometry is a particular example of a set of objects to be called points and set of objects to be called lines which satisfy the axioms of the geometry. Models are useful for picturing the objects of our geometry, but we must be careful to avoid referring to a particular model in our proofs.

David Hilbert (1862 - 1943) presented the first complete, rigorous (by modern standards) development of Euclidean geometry in his book *Grundlagen der Geometrie*, published in 1899. Hilbert fully understood the importance of the fundamental objects of geometry being undefined; as he said, “One must be able to say at all times – instead of points, straight lines, and planes – tables, chairs, and beer mugs” (*Hilbert–Courant* by Constance Reid, Springer-Verlag, 1986, page 57). Hilbert’s development is a *synthetic* one, that is, one which we do not postulate a correspondence between points on a line and the real numbers. Our development will be *metric* in that we will assume knowledge of the real number system and create a way to measure distance along a line by assuming there is always a one-to-one correspondence between points on a line and the real numbers. Hilbert’s system contains many more axioms than ours will. For example, he requires numerous axioms to capture the idea of *betweenness*, whereas we will get this notion from properties of real numbers.