Mathematics 22: Lecture 3

Pure Time Equations and Equations of Motion

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- Example: The general solution of $\frac{du}{dt} = t\sin(t^2)$ is

$$u(t) = \int t\sin(t^2)dt = -\frac{1}{2}\cos(t^2) + c.$$

Initial value problems

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- If follows that a particular solution for

$$\begin{cases} \dfrac{du}{dt} & = g(t), \\ u(t_0) & = u_0, \end{cases}$$

  is

$$u(t) = \int_{t_0}^t g(s)ds + u_0.$$

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$$u(t) = \int_{\pi}^{t} \sin(4s) ds + 10 = -\frac{1}{4}\cos(4s)\Big|_{\pi}^{t} + 10 = -\frac{1}{4}\cos(4t) + \frac{41}{4}.$$

Example

- Consider the initial value problem, for a fixed constant $g > 0$,

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- Note: This is the formula for the height above the ground of an object in free fall near the surface, where g is the acceleration due to gravity (-9.8 meters per second on earth).

Using Maxima

- Maxima commands to solve a first-order initial value problem:
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- Maxima commands to solve a second-order initial value problem:
  - `ode2('diff(u,t,2) = -g, u, t);`
  - `ic2(%, t=0, u=u_0, 'diff(u,t)=v_0);`

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- To define a domain for a variable:
    - `assume(t > 1)`

Newton's second law

- If an object of mass m moves along a straight line, it's position at time t given by $x(t)$, then

  $$m\ddot{x} = F(t, x, \dot{x}),$$

  where F is the force acting on the object (assumed to be a function of time, position, and velocity).

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- Hence, as we saw earlier,

$$x = -\frac{1}{2}gt^2 + v_0 t + h$$

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- If we let $v = \dot{x}$, then we may rewrite this second-order equation as a first-order equation:

$$\dot{v} = g - \frac{a}{m}v^2, v(0) = v_0.$$

Viscous fluids (cont'd)

- Note: $\dot{v} < 0$ if and only if $v > \sqrt{\frac{mg}{a}}$ and $\dot{v} > 0$ if and only if $v < \sqrt{\frac{mg}{a}}$.

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  - If $v_0 < v_T$, then v increases toward v_T.
  - If $v_0 = v_T$, then $v(t) = v_T$ for all $t \geq 0$.
- We call v_T the *terminal velocity* of the object and we call $v \equiv v_T$ an *equilibrium*, or *steady-state*, solution.

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- Then
$$m\ddot{x} = F_r + F_s = -c\dot{x} - kx.$$
- Hence if the object is released from rest at time $t = 0$ from position x_0, then
$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0, x(0) = x_0, \dot{x}(0) = 0.$$

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  - So the potential energy is $mgl(1 - cos(\theta))$.

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- Thus we have, for some constant E,

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- Differentiating both sides with respect to t, we have

$$l\dot{\theta}\ddot{\theta} + g\sin(\theta)\dot{\theta} = 0.$$

- Hence we have the equation for the motion of a pendulum:

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta).$$