Mathematics 22: Lecture 13

Constant-coefficient Equations

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$$\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0$$

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- Then

$$\frac{du}{dt} = \lambda e^{\lambda t} \text{ and } \frac{d^2u}{dt^2} = \lambda^2 e^{\lambda t},$$

so we want

$$0 = \lambda^2 e^{rt} + p\lambda e^{\lambda t} + qe^{\lambda t} = e^{\lambda t}(\lambda^2 + p\lambda + q).$$

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so we want

$$0 = \lambda^2 e^{rt} + p\lambda e^{\lambda t} + qe^{\lambda t} = e^{\lambda t}(\lambda^2 + p\lambda + q).$$

- Since $e^{\lambda t} \neq 0$ for all t, we need $\lambda^2 + p\lambda + q = 0$.

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- Using the quadratic formula, the roots of the characteristic equation are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Distinct real roots

- Suppose $p^2 - 4q > 0$. Then

$$\lambda_1 = \frac{-p - \sqrt{p^2 - 4q}}{2} \text{ and } \lambda_2 = \frac{-p + \sqrt{p^2 - 4q}}{2}$$

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  are distinct real roots of $\lambda^2 + \lambda p + q = 0$.

- It follows that

$$u_1(t) = e^{\lambda_1 t} \text{ and } u_2(t) = e^{\lambda_2 t}$$

  are both solutions of the differential equation.

Distinct real roots (cont'd)

- Since
$$W(t) = \begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{vmatrix} = (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) t},$$
u_1 and u_2 are linearly independent provided $\lambda_1 \neq \lambda_2$.

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  u_1 and u_2 are linearly independent provided $\lambda_1 \neq \lambda_2$.

- Hence, if λ_1 and λ_2 are distinct real roots of the characteristic equation, then

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

  is the general solution of the differential equation.

Example

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- General solution:

$$u(t) = c_1 e^{-3t} + c_2 e^{2t}.$$

Example (cont'd)

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- Hence $c_1 = \frac{3}{5}$ and $c_2 = \frac{7}{5}$.
- So the desired particular solution is

$$u = \frac{3}{5}e^{-3t} + \frac{7}{5}e^{2t}.$$

Double roots

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- If $p^2 - 4q = 0$, then the characteristic equation has a single root

$$\lambda = -\frac{p}{2},$$

which gives us the single solution $u_1(t) = e^{\lambda t}$.

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  - We first note that

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and

$$\frac{d^2u_2}{dt^2} = \lambda^2 te^{\lambda t} + \lambda e^{\lambda t} + \lambda e^{\lambda t} = \left(\frac{p^2}{4}t - p\right) e^{-\frac{p}{2}t}.$$

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  - Hence

$$\begin{aligned}\frac{d^2u_2}{dt^2} + p\frac{du_2}{dt} + qu_2 &= \left(\frac{p^2}{4}t - p + p - \frac{p^2}{2}t + qt\right)e^{-\frac{p}{2}} \\ &= \left(q - \frac{p^2}{4}\right)te^{-\frac{p}{2}t} \\ &= -\left(\frac{p^2 - 4q}{4}\right)te^{\frac{p}{2}t} \\ &= 0.\end{aligned}$$

Independence

- Now

$$
\begin{aligned}
W(t) &= \begin{vmatrix} e^{\lambda t} & te^{\lambda t} \\ \lambda e^{\lambda t} & \lambda te^{\lambda t} + e^{\lambda t} \end{vmatrix} \\
&= ((\lambda t + 1) - \lambda t)e^{2\lambda t} \\
&= e^{2\lambda t}.
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- Hence u_1 and u_2 are linearly independent.
- Thus the general solution is $u(t) = c_1 e^{\lambda t} + c_2 te^{\lambda t}$.

Example

- The characteristic equation of

$$\frac{d^2u}{dt^2} - 4\frac{du}{dt} + 4u = 0$$

is

$$0 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

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- Hence the general solution is

$$u = c_1 e^{2t} + c_2 t e^{2t}.$$

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- Then the roots of the characteristic equation are
$$\lambda_1 = \alpha + \beta i \text{ and } \lambda_2 = \alpha - \beta i.$$
- So
$$z_1(t) = e^{\lambda_1 t} = e^{(\alpha + \beta i)t} = e^{\alpha t} e^{i\beta t}$$
and
$$z_2(t) = e^{\lambda_2 t} = e^{(\alpha - \beta i)t} = e^{\alpha t} e^{-i\beta t}$$
are solutions to the differential equation.

Euler's formula

- Euler showed that
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- Proof:

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= 1 + \theta i - \frac{\theta^2}{2} - \frac{\theta^3 i}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5 i}{5!} - \frac{\theta^6}{6!} - \frac{\theta^7 i}{7!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\
&= \cos(\theta) + i\sin(\theta).
\end{aligned}$$

Real solutions

- Hence

$$z_1(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

and

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- Note: If f and g are real-valued functions and $u(t) = f(t) + ig(t)$ is a solution to
$$\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0,$$
then both f and g are solutions to the equation.

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$$u_1(t) = e^{\alpha t}\cos(\beta t) \text{ and } u_2(t) = e^{\alpha t}\sin(\beta t)$$
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- Note: It may be shown that u_1 and u_2 are linearly independent.
- Hence the general solution is
$$u(t) = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t)).$$

Alternative form

- Note:

$$\begin{aligned} c_1 \cos(\beta t) + c_2 \sin(\beta t) \\ &= \sqrt{c_1^2 + c_2^2}\left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\cos(\beta t) + \frac{c_2}{\sqrt{c_1^2 + c_2^2}}\sin(\beta t)\right) \\ &= A(\cos(\phi)\cos(\beta t) + \sin(\phi)\sin(\beta t)) \\ &= A\cos(\beta t - \phi), \end{aligned}$$

where

$$A = \sqrt{c_1^2 + c_2^2} \text{ and } \tan(\phi) = \frac{c_2}{c_1}.$$

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- Hence the general solution may be written as $u(t) = Ae^{\alpha t}\cos(\beta t - \phi)$.
- This is the *phase-amplitude* form of the general solution.

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$$\lambda_2 = \frac{-1 + \sqrt{1-4}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$
- Hence the general solution is
$$u(t) = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

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- So the solution of the initial-value problem is

$$u(t) = A\cos(kt).$$

Example: motion of a pendulum

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- Linearized equation: For small values of θ, $\ddot{\theta} = -\frac{g}{l}\theta$
- From the previous example, with initial conditions $\theta(0) = A$ and $\dot{\theta}(0) = 0$,

$$\theta = A\cos\left(\sqrt{\frac{g}{l}}t\right).$$