Lecture 9: Motion Along a Curve

9.1 Components of velocity and acceleration

Suppose $f(t)$ is the position of a particle moving along a curve C in $\mathbb{R}^n$, where $f : \mathbb{R} \to \mathbb{R}^n$, and let $\mathbf{v}(t) = f'(t)$ and $\mathbf{a}(t) = f''(t)$ be the velocity and acceleration of the particle at time t. Now we may write

$$\mathbf{v}(t) = |\mathbf{v}(t)|T(t),$$

where $T(t)$ is the unit tangent vector at $f(t)$, so

$$\begin{aligned}\mathbf{a}(t) &= |\mathbf{v}(t)|T'(t) + \frac{d}{dt}|\mathbf{v}(t)|T(t)\\ &= |\mathbf{v}(t)||T'(t)|N(t) + \frac{d}{dt}|\mathbf{v}(t)|T(t)\\ &= |\mathbf{v}(t)|^2\kappa(t)N(t) + \frac{d}{dt}|\mathbf{v}(t)|T(t),\end{aligned}$$

where $N(t)$ is the principal unit normal vector at $f(t)$ and $\kappa(t)$ is the curvature of C at $f(t)$. In other words, we may write

$$\mathbf{a}(t) = a_T T(t) + a_N N(t),$$

where

$$a_T = \frac{d}{dt}|\mathbf{v}(t)|$$

and

$$a_N = |\mathbf{v}(t)|^2\kappa(t).$$

In particular, this says that the acceleration of the particle always lies in the plane of the unit tangent and principal unit normal vectors. Moreover,

$$\mathbf{a}(t)\cdot T(t) = a_T(T(t)\cdot T(t)) + a_N(N(t)\cdot T(t)) = a_T$$

and

$$\mathbf{a}(t)\cdot N(t) = a_T(T(t)\cdot N(t)) + a_N(N(t)\cdot N(t)) = a_N,$$

so a_T and a_N are just the components of the acceleration vector in the direction of $T(t)$ and $N(t)$, respectively.

Example Suppose a particle moves along a helix H with position at time t given by $f(t) = (\cos(2\pi t), \sin(2\pi t), t)$. Then the velocity and acceleration of the particle are

$$\mathbf{v}(t) = (-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1)$$

and

$$\mathbf{a}(t) = (-4\pi^2\cos(2\pi t), -4\pi^2\sin(2\pi t), 0),$$

respectively. The speed of the particle is

$$|\mathbf{v}(t)| = \sqrt{4\pi^2 + 1},$$

the unit tangent vector is

$$T(t) = \frac{1}{\sqrt{4\pi^2+1}}(-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1),$$

$$T'(t) = \frac{1}{\sqrt{4\pi^2+1}}(-4\pi^2\cos(2\pi t), -4\pi^2\sin(2\pi t), 0),$$

and the principal unit normal vector is

$$N(t) = (-\cos(2\pi t), -\sin(2\pi t), 0).$$

Hence the component of acceleration in the direction of $T(t)$ is

$$a_T = \mathbf{a}(t) \cdot T(t) = \frac{1}{\sqrt{4\pi^2+1}}(8\pi^3\cos(2\pi t)\sin(2\pi t) - (8\pi^3\cos(2\pi t)\sin(2\pi t)) = 0$$

and the component of acceleration in the direction of $N(t)$ is

$$a_N = \mathbf{a}(T) \cdot N(t) = 4\pi^2\cos^2(2\pi t) + 4\pi^2\sin^2(2\pi t) = 4\pi^2.$$

Example Suppose a particle moves along an ellipse E with position at time t given by $f(t) = (4\cos(t), 2\sin(t))$. Then the velocity and acceleration of the particle are

$$\mathbf{v}(t) = (-4\sin(t), 2\cos(t))$$

and

$$\mathbf{a}(t) = (-4\cos(t), -2\sin(t)),$$

respectively. The speed of the particle is

$$|\mathbf{v}(t)| = \sqrt{16\sin^2(t) + 4\cos^2(t)} = \sqrt{12\sin^2(t) + 4}$$

and the curvature of E is

$$\kappa(t) = \frac{8}{(12\sin^2(t)+4)^{\frac{3}{2}}}.$$

Hence the component of acceleration in the direction of the the unit tangent vector $T(t)$ is

$$a_T = \frac{d}{dt}|\mathbf{v}(t)| = \frac{12\sin(t)\cos(t)}{\sqrt{12\sin^2(t)+4)}}$$

and the component of acceleration in the direction of the principal unit normal vector $N(t)$ is

$$a_N = |\mathbf{v}(t)|^2\kappa(t) = (12\sin^2(t)+4)\frac{8}{(12\sin^2(t)+4)^{\frac{3}{2}}} = \frac{8}{\sqrt{12\sin^2(t)+4}}.$$

Note that when $t = 0$ or $t = \pi$, $a_T = 0$ and $a_N = 4$, and when $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$, $a_T = 0$ and $a_N = 2$.