Lecture 8: Arc Length and Curvature

8.1 Arc Length

Suppose $f : \mathbb{R} \to \mathbb{R}^n$ parametrizes the curve C in $\mathbb{R}^n$ as t goes from a to b. If we think of an object moving along C so that its position at time t is $f(t)$, then the speed of the object at time t is $|f'(t)|$. Let L be the length of C. If we divide $[a, b]$ into n equal intervals of length

$$\Delta t = \frac{b-a}{n}$$

with endpoints $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, then

$$\sum_{k=0}^{n-1} |f'(t_k)|\Delta t$$

provides an approximation of L which should improve as n increases. Indeed, if f' is continuous we should have

$$L = \lim_{n\to\infty} \sum_{k=0}^{n-1} |f'(t_k)|\Delta t = \int_a^b |f'(t)|dt.$$

Example The function $f(t) = (\cos(2\pi t), \sin(2\pi t), t)$ parametrizes one loop of a helix H as t goes from 0 to 1. If L is the length of this part of H, then

$$\begin{aligned} L &= \int_0^1 |f'(t)|dt \\ &= \int_0^1 \sqrt{4\pi^2 \sin^2(2\pi t) + 4\pi^2 \cos^2(2\pi t) + 1}\; dt \\ &= \int_0^1 \sqrt{4\pi^2 + 1}\; dt \\ &= \sqrt{4\pi^2 + 1}. \end{aligned}$$

Suppose C is the graph of a function $g : \mathbb{R} \to \mathbb{R}$ over an interval $[a, b]$. We may parametrize C with $f(t) = (t, g(t))$, $a \le t \le b$. Hence if L is the length of C, then

$$L = \int_a^b \sqrt{1 + (g'(t))^2}\; dt.$$

Example The length of the curve $y = x^{\frac{3}{2}}$ over the interval $[0, 1]$ is

$$\int_0^1 \sqrt{1 + \frac{9}{4}x}\; dx = \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}} \Big|_0^1 = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{\frac{3}{2}} - 1\right) = 1.43971,$$

where the final value has been rounded to four decimal places.

Example We may parametrize a circle of radius $r > 0$ with center at the origin with $f(t) = (r\cos(t), r\sin(t))$. Then the circumference of the circle is

$$C = \int_0^{2\pi} \sqrt{r^2\sin^2(t) + r^2\cos^2(t)}\, dt = \int_0^{2\pi} r dt = 2\pi r.$$

Note that

$$\int_0^{4\pi} \sqrt{r^2\sin^2(t) + r^2\cos^2(t)}\, dt = \int_0^{4\pi} r dt = 4\pi r$$

because $f(t)$ traverses the circle twice as t goes from 0 to 4π.

8.2 Normal and binormal vectors

Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a smooth parametrization of a curve C and $T(t)$ is the unit tangent vector to C at $f(t)$. Then

$$T(t) \cdot T(t) = 1$$

for all t, so

$$\frac{d}{dt}(T(t) \cdot T(t)) = 0.$$

But

$$\frac{d}{dt}(T(t) \cdot T(t)) = T(t) \cdot T'(t) + T'(t)T(t) = 2T(t) \cdot T'(t).$$

Hence $T(t) \cdot T'(t) = 0$; that is, $T(t)$ and $T'(t)$ are orthogonal.

Definition Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a smooth parametrization of a curve C and $T(t)$ is the unit tangent vector to C at $f(t)$. We call

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

the *principal unit normal vector* to C at $f(t)$. If $n = 3$, we call

$$B(t) = T(t) \times N(t)$$

the *binormal vector* to C at $f(t)$.

Note that $B(t)$ is a unit vector orthogonal to both $T(t)$ and $N(t)$.

Example Consider $f(t) = (\cos(2\pi t), \sin(2\pi t), t)$, a smooth parametrization of a circular helix H. Then

$$f'(t) = (-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1)$$

and

$$|f'(t)| = \sqrt{4\pi^2\sin^2(2\pi t) + 4\pi^2\cos^2(2\pi t) + 1} = \sqrt{4\pi^2 + 1},$$

so the unit tangent vector is

$$T(t) = \frac{1}{\sqrt{4\pi^2+1}}(-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1).$$

Then

$$T'(t) = \frac{1}{\sqrt{4\pi^2+1}}(-4\pi^2\cos(2\pi t), -4\pi^2\sin(2\pi t), 0)$$

and

$$|T'(t)| = \frac{1}{\sqrt{4\pi^2+1}}\sqrt{16\pi^4\cos^2(2\pi t) + 16\pi^4\sin^2(2\pi t)} = \frac{4\pi^2}{\sqrt{4\pi^2+1}},$$

so the principal unit normal vector is

$$N(t) = (-\cos(2\pi t), -\sin(2\pi t), 0).$$

Finally, the binormal vector is

$$B(t) = T(t) \times N(t) = \frac{1}{\sqrt{4\pi^2+1}}(\sin(2\pi t), -\cos(2\pi t), 2\pi).$$

For example, at $t = 0$ we have

$$f(0) = (1, 0, 0),$$

$$T(0) = \frac{1}{\sqrt{4\pi^2+1}}(0, 2\pi, 1),$$

$$N(0) = (-1, 0, 0),$$

and

$$B(0) = \frac{1}{\sqrt{4\pi^2+1}}(0, -1, 2\pi).$$

See the figure below.

8.3 Curvature

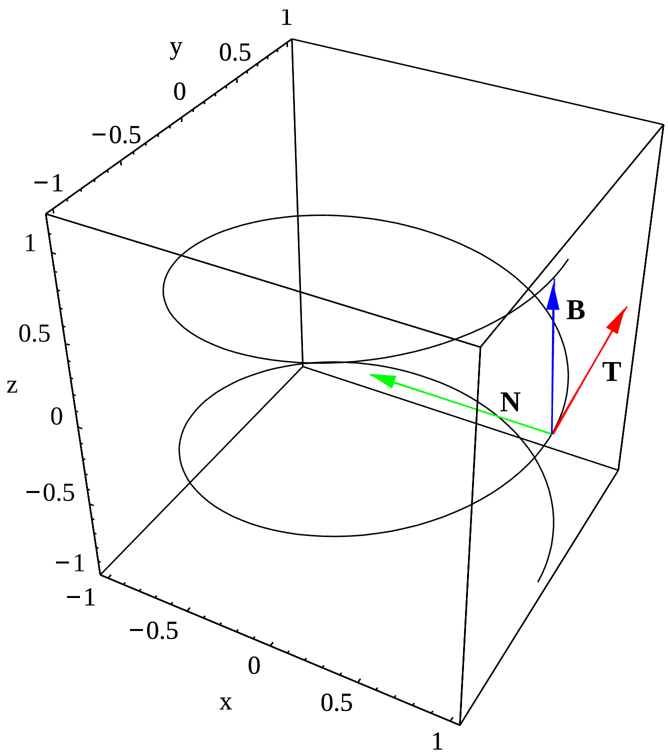
Definition Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a smooth parametrization of a curve C in $\mathbb{R}^n$. We call

$$\kappa(t) = \frac{|T'(t)|}{|f'(t)|}$$

the *curvature* of C at $f(t)$.

Example Consider the parametrization $f(t) = (r\cos(t), r\sin(t))$ of a circle C of radius $r > 0$ centered at $(0, 0)$. Then

$$f'(t) = (-r\sin(t), r\cos(t)),$$



A helix with unit tangent, normal, and binormal vectors

$$|f'(t)| = r,$$

$$T(t) = (-\sin(t), \cos(t)),$$

$$T'(t) = (-\cos(t), -\sin(t)),$$

and

$$|T'(t)| = 1.$$

Hence the curvature of C is

$$\kappa(t) = \frac{1}{r}.$$

Note that $\kappa(t)$ is a constant and $\lim_{r\to\infty} \kappa(t) = 0$.

Consider a smooth parametrization $f : \mathbb{R} \to \mathbb{R}^3$ of a curve C in $\mathbb{R}^3$. Let

$$\mathbf{v}(t) = f'(t),$$

the *velocity vector*, and

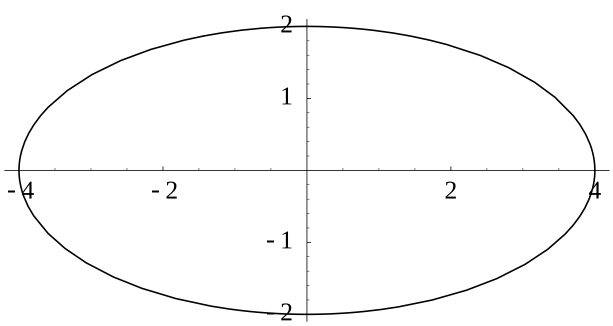
$$\mathbf{a}(t) = f''(t),$$

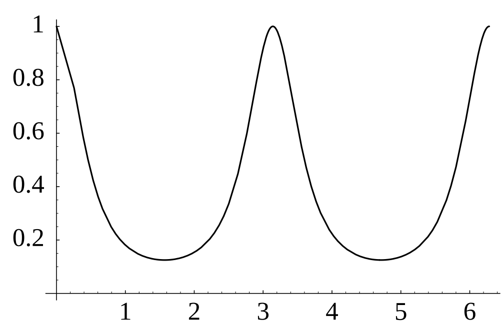
the *acceleration vector*. If $T(t)$ is the unit tangent vector, then

$$\mathbf{v}(t) = |\mathbf{v}(t)|T(t).$$

Hence
$$\mathbf{a}(t) = |\mathbf{v}(t)|T'(t) + T(t)\frac{d}{dt}|\mathbf{v}(t)|,$$
and so
$$\mathbf{v}(t) \times \mathbf{a}(t) = |\mathbf{v}(t)|^2(T(t) \times T'(t)) + |\mathbf{v}(t)|\frac{d}{dt}|\mathbf{v}(t)|(T(t) \times T(t)) = |\mathbf{v}(t)|^2(T(t) \times T'(t)).$$
Thus
$$|\mathbf{v}(t) \times \mathbf{a}(t)| = |\mathbf{v}(t)|^2|T(t)||T'(t)| = |\mathbf{v}(t)|^2|T'(t)|,$$
and so
$$\kappa(t) = \frac{|T'(t)|}{|\mathbf{v}(t)|} = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}.$$

Example Consider the ellipse E parametrized by $f(t) = (4\cos(t), 2\sin(t), 0)$. Then
$$\mathbf{v}(t) = (-4\sin(t), 2\cos(t), 0),$$
$$\mathbf{a}(t) = (-4\cos(t), -2\sin(t), 0),$$
and
$$\mathbf{v}(t) \times \mathbf{a}(t) = (0, 0, 8\sin^2(t) + 8\cos^2(t)) = (0, 0, 8),$$
so
$$\kappa(t) = \frac{8}{(16\sin^2(t) + 4\cos^2(t))^{\frac{3}{2}}} = \frac{8}{(12\sin^2(t) + 4)^{\frac{3}{2}}}.$$
For example, $\kappa(0) = 1$ and $\kappa\left(\frac{\pi}{2}\right) = \frac{1}{8}$.





The ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$ and its curvature