Lecture 7: Derivatives of Functions from $\mathbb{R}$ to $\mathbb{R}^n$

7.1 Derivatives

Recall that the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point t is

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

provided the limit exists.

Definition The *derivative* of a function $f : \mathbb{R} \to \mathbb{R}^n$ at a point t is

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

provided the limit exists.

If $n = 1$, $f'(t)$ is the slope of the line tangent to the graph of f at $(t, f(t))$. For $n > 1$, $f'(t)$ is the vector tangent to the curve parametrized by f at $f(t)$.

Suppose $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$. Then

$$\begin{aligned}
f'(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\
&= \lim_{h \to 0} \left(\frac{f_1(t+h) - f_1(t)}{h}, \frac{f_2(t+h) - f_2(t)}{h}, \ldots, \frac{f_n(t+h) - f_n(t)}{h} \right) \\
&= \left(\lim_{h \to 0} \frac{f_1(t+h) - f_1(t)}{h}, \lim_{h \to 0} \frac{f_2(t+h) - f_2(t)}{h}, \ldots, \lim_{h \to 0} \frac{f_n(t+h) - f_n(t)}{h} \right) \\
&= (f_1'(t), f_2'(t), \ldots, f_n'(t)).
\end{aligned}$$

Proposition If $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable at t and $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$, then

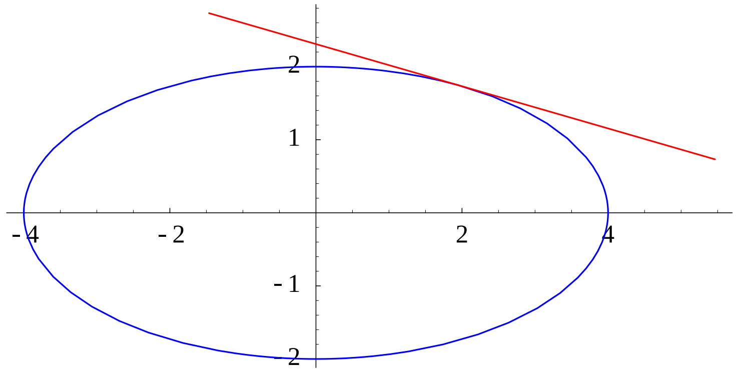
$$f'(t) = (f_1'(t), f_2'(t), \ldots, f_n'(t)).$$

Suppose $f : \mathbb{R} \to \mathbb{R}^n$ parametrizes a curve C. If f is differentiable at a and $f'(a) \neq \mathbf{0}$, then an equation for the line tangent to C at a is

$$\mathbf{x} = f(a) + f'(a)(t - a).$$

Example Let $f(t) = (4\cos(t), 2\sin(t))$. Then

$$f'(t) = (-4\sin(t), 2\cos(t)).$$



Ellipse with tangent line

For example,

$$f'\left(\frac{\pi}{3}\right) = (-2\sqrt{3}, 1).$$

Note that f parametrizes the ellipse E with equation

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

The equation of the line L which is tangent to E at $f\left(\frac{\pi}{3}\right) = (2, \sqrt{3})$ is

$$\mathbf{x} = (2, \sqrt{3}) + (-2\sqrt{3}, 1)\left(t - \frac{\pi}{3}\right).$$

Equivalently, the parametric equations of L are

$$x = 2 - 2\sqrt{3}\left(t - \frac{\pi}{3}\right),$$
$$y = \sqrt{3} + \left(t - \frac{\pi}{3}\right).$$

Example If $f(t) = (\cos(2\pi t), \sin(2\pi t), t)$, then

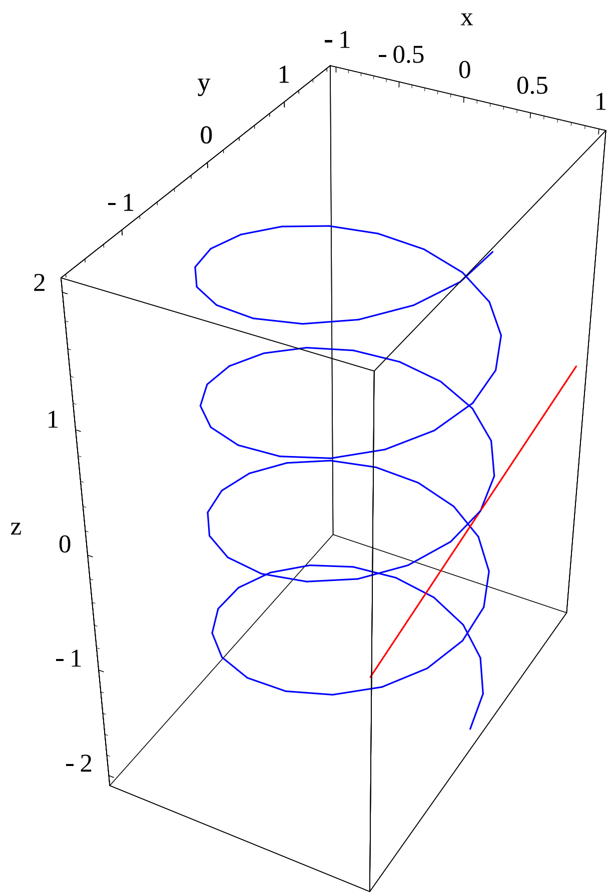
$$f'(t) = (-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1).$$

Hence the equation of the line tangent to the helix C parametrized by f at $f(0) = (1, 0, 0)$ has vector equation

$$\mathbf{x} = (1, 0, 0) + (0, 2\pi, 1)t$$

and parametric equations

$$x = 1,$$
$$y = 2\pi t,$$
$$z = t.$$



Helix with tangent line

Example Let $f(t) = (t^2, t^4)$. Then

$$f'(t) = (2t, 4t^3),$$

and so

$$f'(0) = (0, 0).$$

Definition Suppose $f : \mathbb{R} \to \mathbb{R}^n$ parametrizes a curve C for t in some open interval (a, b). If f' is continuous and $f'(t) \neq \mathbf{0}$ for all t in (a, b), then we call f a *smooth* parametrization of C.

Example $f(t) = (t, t^2)$ is a smooth parametrization of the parabola $y = x^2$, while $g(t) = (t^3, t^6)$ is not a smooth parametrization of the same parabola. However, we would say the g is a *piecewise smooth* parametrization of $y = x^2$.

Definition If $f : \mathbb{R} \to \mathbb{R}^n$ parametrizes a curve C and $f'(a) \neq 0$, we call

$$T(a) = \frac{f'(a)}{|f'(a)|}$$

the *unit tangent vector* to C at $f(a)$.

Example If $f(t) = (4\cos(t), 2\sin(t))$, then we saw above that

$$f'\left(\frac{\pi}{3}\right) = (-2\sqrt{3}, 1).$$

Thus

$$\left|f'\left(\frac{\pi}{3}\right)\right| = \sqrt{12+1} = \sqrt{13},$$

so the unit tangent vector at $(2, \sqrt{3})$ is

$$T\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{13}}(-2\sqrt{3}, 1).$$

We will also use the Leibniz notation to denote derivatives. That is, we let

$$\frac{d}{dt}f(t) = f'(t).$$

Note that our author will sometimes also denote $f'(t)$ by $D_t f(t)$.

Proposition Suppose $f : \mathbb{R} \to \mathbb{R}^n$, $g : \mathbb{R} \to \mathbb{R}^n$, $\varphi : \mathbb{R} \to \mathbb{R}$ are all differentiable and c is a scalar. Then

$$\frac{d}{dt}(f(t) + g(t)) = \frac{d}{dt}f(t) + \frac{d}{dt}g(t),$$

$$\frac{d}{dt}cf(t) = c\frac{d}{dt}f(t),$$

$$\frac{d}{dt}(\varphi(t)f(t)) = \varphi(t)f'(t) + \varphi'(t)f(t),$$

$$\frac{d}{dt}(f(\varphi(t)) = f'(\varphi(t))\varphi'(t),$$

$$\frac{d}{dt}(f(t)\cdot g(t)) = f(t)\cdot g'(t) + f'(t)\cdot g(t),$$

$$\frac{d}{dt}(f(t)\times g(t)) = f(t)\times g'(t) + f'(t)\times g(t).$$

Proof We will prove the product rule for dot products. If $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ and $g(t) = (g_1(t), g_2(t), \ldots, g_n(t))$, then

$$\begin{aligned}
\frac{d}{dt}(f(t)\cdot g(t)) &= \frac{d}{dt}(f_1(t)g_1(t) + f_2(t)g_2(t) + \cdots + f_n(t)g_n(t)) \\
&= (f_1(t)g_1'(t) + f_1'(t)g_1(t), f_2(t)g_2'(t) + f_2'(t)g_2(t), \ldots, \\
&\qquad f_n(t)g_n'(t) + f_n'(t)g_n(t)) \\
&= (f_1(t)g_1'(t) + f_2(t)g_2'(t), \ldots, f_n(t)g_n'(t)) \\
&\qquad + (f_1'(t)g_1(t), f_2'(t)g_2(t), \ldots, f_n'(t)g_n(t)) \\
&= f(t)\cdot g'(t) + f'(t)\cdot g(t).
\end{aligned}$$

7.2 Integrals

Definition If $f : \mathbb{R} \to \mathbb{R}^n$ with $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ and each of $f_1, f_2, \ldots, f_n$ is integrable on $[a, b]$, then the *definite integral* of f on the interval $[a, b]$ is

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, \int_a^b f_2(t), \ldots, \int_a^b f_n(t)dt\right).$$

Similarly, we define the *indefinite integral* of f by

$$\int f(t)dt = \left(\int f_1(t)dt, \int f_2(t)dt, \ldots, \int f_n(t)dt\right).$$

Example If $f(t) = (t^2, \sin(2\pi t))$, then

$$\int_0^1 f(t)dt = \left(\int_0^1 t^2 dt, \int_0^1 \sin(2\pi t)dt\right) = \left(\frac{1}{3}, 0\right).$$

Example If $f(t) = (t, t^2, t^3)$, then

$$\int f(t)dt = \left(\frac{1}{2}\,t^2, \frac{1}{3}\,t^3, \frac{1}{4}\,t^4\right) + \mathbf{c}.$$

Note that $\mathbf{c}$, the constant of integration, is a vector.