Lecture 6: Functions from $\mathbb{R}$ to $\mathbb{R}^n$

6.1 Some terminology

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the set of all points $\mathbf{x}$ in $\mathbb{R}^n$ satisfying $\mathbf{x} = f(t)$ for some t in the domain of f is called a *curve*. Note that $f(t)$ is a vector in $\mathbb{R}^n$, so we may define functions $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, n$, such that

$$f_k(t) = \text{ kth coordinate of } f(t).$$

That is,

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t)).$$

The functions $f_1, f_2, \ldots, f_n$ are the *component* functions of f. If C is the curve determined by f, we call $\mathbf{x} = f(t)$ the *vector equation* of C and, writing $\mathbf{x} = (x_1, x_2, \ldots, x_n)$,

$$\begin{aligned} x_1 &= f_1(t), \\ x_2 &= f_2(t), \\ &\vdots \\ x_n &= f_n(t), \end{aligned}$$

the *parametric equations* of C.

Example The function

$$f(t) = (\cos(t), \sin(t))$$

parametrizes the unit circle C with center at $(0, 0)$ in $\mathbb{R}^2$. We may write the parametric equations as

$$\begin{aligned} x &= \cos(t), \\ y &= \sin(t). \end{aligned}$$

Note that $f(t)$ traverses C in the counterclockwise direction once over every interval of length 2π.

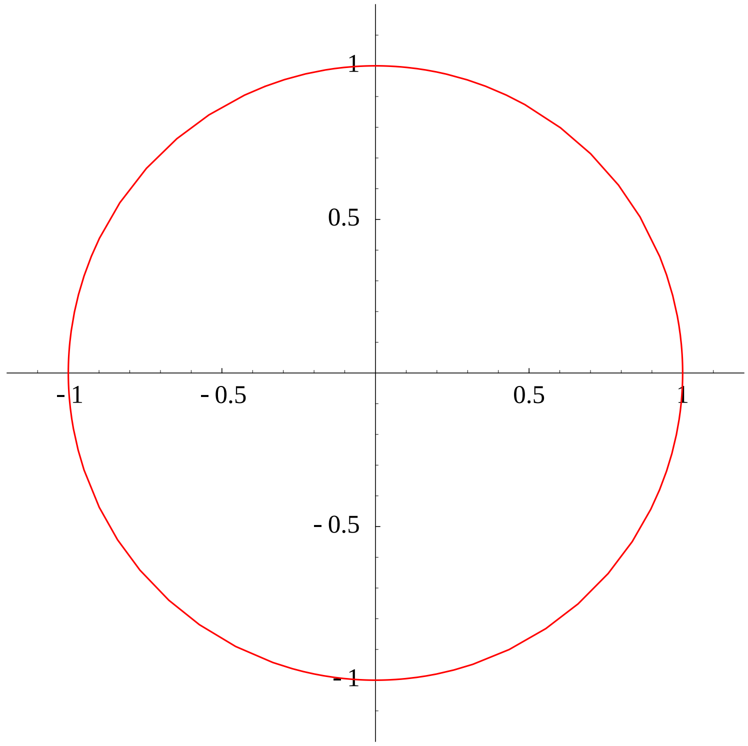
Example Note that the function

$$g(t) = (\cos(2\pi t), \sin(2\pi t))$$

also parametrizes the unit circle with center at $(0, 0)$ in $\mathbb{R}^2$. However, $g(t)$ traverses the circle in the counterclockwise direction once over every interval of length 1.

Example Note that the function

$$h(t) = (\sin(2\pi t), \cos(2\pi t))$$



Unit circle parametrized by $f(t) = (\cos(t), \sin(t))$

also parametrizes the unit circle with center at $(0,0)$ in $\mathbb{R}^2$. However, $h(t)$ traverses the circle in the clockwise direction once over every interval of length 1.

Example The function

$$f(t) = (\cos(2\pi t), \sin(2\pi t), t)$$

parametrizes a helix which wraps around the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

Example The function

$$f(t) = (t, t^2)$$

parametrizes the parabola $y = x^2$.

Example The function

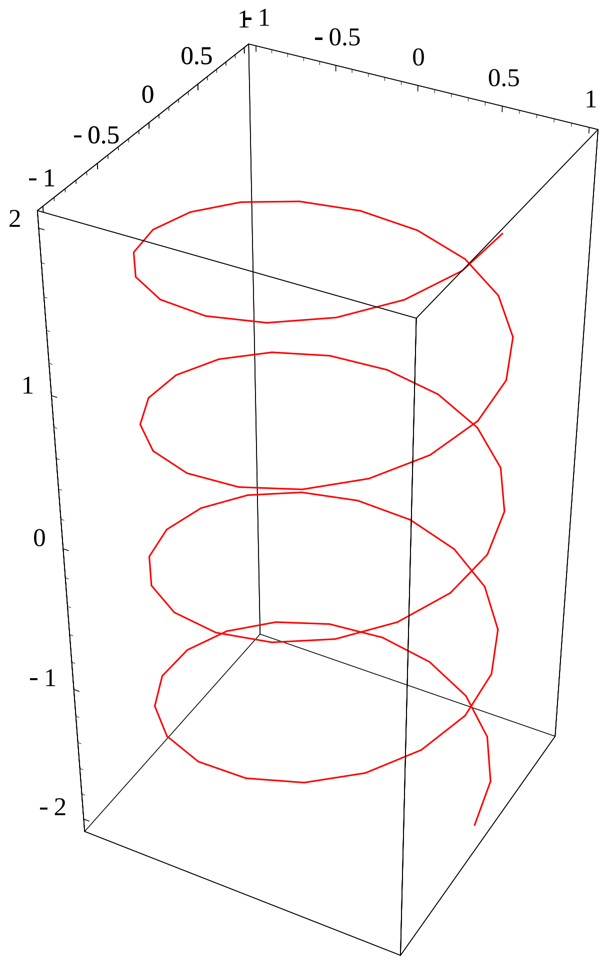
$$g(t) = (t^2, t^4)$$

parametrizes the parabola $Y = x^2$ with $x \geq 0$. Note that $g(t)$ approaches $(0,0)$ from the right as t goes from $-\infty$ to ∞, and then moves away from $(0,0)$ as t goes from 0 to ∞.

6.2 Limits and continuity

Definition Given a function $f : \mathbb{R} \to \mathbb{R}^n$, we say that the *limit* of $f(t)$ as t approaches a is **L**, denoted

$$\lim_{t \to c} f(t) = L,$$



Helix parametrized by $f(t) = (\cos(2\pi t), \sin(2\pi t), t)$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(t) - \mathbf{L}| < \epsilon$$

whenever

$$0 < |t - a| < \delta.$$

Note that if $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ and $\mathbf{L} = (L_1, L_2, \ldots, L_n)$, then

$$|f(t) - L| = \sqrt{(f_1(t) - L_1)^2 + (f_2(t) - L_2)^2 + \cdots + (f_n(t) - L_n)^2}.$$

Hence $|f(t) - \mathbf{L}|$ is "small" if and only if $|f_k(t) - L_k|$ is "small" for $k = 1, 2, \ldots, n$. Consequently,

$$\lim_{t \to a} f(t) = \mathbf{L}$$

if and only if

$$\lim_{t \to a} f_k(t) = L_k$$

for $k = 1, 2, \ldots, n$. That is,

$$\lim_{t \to a} f(t) = \left(\lim_{t \to a} f_1(t), \lim_{t \to a} f_2(t), \ldots, \lim_{t \to a} f_n(t) \right).$$

Example $\lim_{t \to \pi} (\cos(t), \sin(t), t^2) = (-1, 0, \pi^2)$.

Note that limits from the left and right may be defined analogous to the definitions for functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *continuous* at a point a in $\mathbb{R}$ if $\lim_{t \to a} f(t) = f(a)$.

Proposition If $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$, then f is continuous at a if and only if $f_1, f_2, \ldots, f_n$ are each continuous at a.

Note that continuous from the right and continuous from the left may be defined analogous to the definitions for functions from $\mathbb{R}$ to $\mathbb{R}$.

Example $f(t) = (\cos(t), \sin(3\pi t), \sqrt{t})$ is continuous on $[0, \infty)$. Recall that this means f is continuous for all points t in $(0, \infty)$ and f is continuous from the right at 0.