Lecture 5: Lines and Planes

5.1 Lines in $\mathbb{R}^n$

Definition Given vectors $\mathbf{p}$ and $\mathbf{v}$ in $\mathbb{R}^n$, with $\mathbf{v} \neq \mathbf{0}$, the set of all vectors $\mathbf{x}$ satisfying

$$\mathbf{x} = \mathbf{p} + t\mathbf{v},$$

$-\infty < t < \infty$, is called a *line*.

If $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then the *vector equation*

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

is equivalent to the *parametric equations*

$$\begin{aligned} x_1 &= p_1 + v_1 t, \\ x_2 &= p_2 + v_2 t, \\ &\vdots \\ x_n &= p_n + v_n t. \end{aligned}$$

Example Let L be the line through the points $\mathbf{p} = (1, 2, 1)$ and $\mathbf{q} = (-1, 4, 4)$ in $\mathbb{R}^3$. If we let

$$\mathbf{v} = \mathbf{q} - \mathbf{p} = (-2, 2, 3),$$

then L has vector equation

$$\mathbf{x} = (1, 2, 1) + t(-2, 2, 3)$$

and parametric equations

$$\begin{aligned} x &= 1 - 2t, \\ y &= 2 + 2t, \\ z &= 1 + 3t. \end{aligned}$$

Definition We say lines L_1 and L_2 with vector equations $\mathbf{x} = \mathbf{p}_1 + t\mathbf{v}_1$ and $\mathbf{x} = \mathbf{p}_2 + t\mathbf{v}_2$ are *parallel* if $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel.

5.2 Lines in $\mathbb{R}^2$

Suppose L is a line in the plane with parametric equations

$$\begin{aligned} x &= \alpha + \beta t, \\ y &= \gamma + \delta t. \end{aligned}$$

Multiplying the first equation by δ and the second by β, we have

$$\delta x = \alpha\delta + \beta\delta t,$$
$$\beta y = \beta\gamma + \beta\delta t.$$

Subtracting the second from the first gives us

$$\delta x - \beta y = \alpha\delta - \beta\gamma.$$

If we let $a = \delta$, $b = -\beta$, and $c = \beta\gamma - \alpha\delta$, then we have

$$ax + by + c = 0,$$

a familiar form for the equation of a line in $\mathbb{R}^2$, which we call the *scalar* form of the equation of L. Now let $\mathbf{n} = (a, b)$ and suppose (x_0, y_0) is a point on L. Then $c = -ax_0 - by_0$, so we may rewrite the previous equation as

$$\mathbf{n} \cdot (x, y) - \mathbf{n} \cdot (x_0, y_0) = 0,$$

or, equivalently,

$$\mathbf{n} \cdot (x - x_0, y - y_0) = 0.$$

Hence, geometrically, we may think of L as the set of all points (x, y) in the plane such that the vector from (x, y) to (x_0, y_0) is orthogonal to $\mathbf{n}$.

Example Suppose

$$4x - 3y + 1 = 0$$

is a scalar equation for the line L in $\mathbb{R}^2$. Let $\mathbf{n} = (4, -3)$ and $\mathbf{p} = (-1, -1)$. Then $\mathbf{p}$ is a point on L, $\mathbf{n}$ is a normal vector for L, and we could write the scalar equation for L as

$$(4, -3) \cdot (x + 1, y + 1) = 0.$$

5.3 Planes in $\mathbb{R}^n$

Definition Given vectors $\mathbf{p}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$, with $\mathbf{v} \neq \mathbf{0}$, $\mathbf{w} \neq \mathbf{0}$, and $\mathbf{v}$ and $\mathbf{w}$ not parallel, the set of all points $\mathbf{x}$ satisfying

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} + s\mathbf{w},$$

$-\infty < t < \infty$, $-\infty < s < \infty$, is called a *plane.*

If $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then the *vector equation*

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} + s\mathbf{w}$$

is equivalent to the *parametric equations*

$$\begin{aligned} x_1 &= p_1 + v_1 t + w_1 s, \\ x_2 &= p_2 + v_2 t + w_2 s, \\ &\vdots \\ x_n &= p_n + v_n t + w_n s. \end{aligned}$$

Example Let P be the plane in $\mathbb{R}^3$ through the points $\mathbf{p} = (1, 1, 2)$, $\mathbf{q} = (2, -2, 3)$, and $\mathbf{r} = (-2, 3, 4)$. If we let

$$\mathbf{v} = \mathbf{q} - \mathbf{p} = (1, -3, 1)$$

and

$$\mathbf{w} = \mathbf{r} - \mathbf{p} = (-3, 2, 2),$$

then P has vector equation

$$\mathbf{x} = (1, 1, 2) + t(1, -3, 1) + s(-3, 2, 2)$$

and parametric equations

$$\begin{aligned} x &= 1 + t - 3s, \\ y &= 1 - 3t + 2s, \\ z &= 2 + t + 2s. \end{aligned}$$

5.4 Planes in $\mathbb{R}^3$

Given parametric equations for a plane P in $\mathbb{R}^3$, we could eliminate, in a manner similar to what we did for lines in $\mathbb{R}^2$, the t and s variables and arrive at a single *scalar equation* for P. That is, for appropriate scalars a, b, c,and d, we may view P as the set of all points (x, y, z) satisfying

$$ax + by + cz + d = 0.$$

Moreover, if $\mathbf{n} = (a, b, c)$ and $\mathbf{p} = (x_0, y_0, z_0)$ is a point on P, then we may rewrite the scalar equation as

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0,$$

or, equivalently,

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

That is, P is the set of all points $\mathbf{x}$ such that $\mathbf{n}$ is orthogonal to $\mathbf{x} - \mathbf{p}$.

Example As in the previous example, let P be the plane in $\mathbb{R}^3$ through the points $\mathbf{p} = (1, 1, 2)$, $\mathbf{q} = (2, -2, 3)$, and $\mathbf{r} = (-2, 3, 4)$. If we let

$$\mathbf{v} = \mathbf{q} - \mathbf{p} = (1, -3, 1)$$

and

$$\mathbf{w} = \mathbf{r} - \mathbf{p} = (-3, 2, 2),$$

then

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = (-8, -5, -7)$$

is a normal vector for P. Hence a scalar equation for P is given by

$$(-8, -5, -7) \cdot (x - 1, y - 1, z - 2) = 0,$$

or, equivalently,

$$8x + 5y + 7z - 27 = 0.$$

Definition Suppose P and Q are planes in R^3 with scalar equations

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

and

$$\mathbf{m} \cdot (\mathbf{x} - \mathbf{q}) = 0,$$

where $\mathbf{n}$ and $\mathbf{m}$ have been chosen so that $\mathbf{n} \cdot \mathbf{m} \geq 0$. We call the smallest positive angle between $\mathbf{n}$ and $\mathbf{m}$ the *angle* between P and Q.

Example Let θ be the angle between the planes P and Q with equations

$$3x - 4y + 2z = 14$$

and

$$x + y - z = 1.$$

Let $\mathbf{n} = (3, -4, 2)$ and $\mathbf{m} = (-1, -1, 1)$. Then

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{29}\sqrt{3}}\right) = \cos^{-1}\left(\sqrt{\frac{3}{29}}\right) = 1.2433,$$

where the final result has been rounded to 4 decimal places.

Let P be a plane with scalar equation $ax + by + cz + d = 0$ and let $\mathbf{q}$ be point in $\mathbb{R}^3$. Let $\mathbf{p}$ be a point on P and let $\mathbf{n} = (a, b, c)$ be a normal vector for P. If D is the distance from $\mathbf{q}$ to P, then

$$\begin{aligned} D &= |\mathrm{proj}_{\mathbf{n}}(\mathbf{q} - \mathbf{p})| \\ &= \left|(\mathbf{q} - \mathbf{p}) \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| \\ &= \frac{|\mathbf{q} \cdot \mathbf{n} - \mathbf{p} \cdot \mathbf{n}|}{|\mathbf{n}|}. \end{aligned}$$

Now $\mathbf{p} \cdot \mathbf{n} = -d$, so, if we let $\mathbf{q} = (x_1, x_2, x_3)$, we have

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example The distance D from the plane with equation $x + y + z = 1$ to the point $(2, 2, 2)$ is

$$D = \frac{2 + 2 + 2 - 1}{\sqrt{3}} = \frac{5}{\sqrt{3}}.$$

Example A similar formula works for finding the distance from a point in the plane to a line. For example, the distance D from the point $(2, 3)$ to the line $2x + y - 4 = 0$ is

$$D = \frac{|4 + 3 - 4|}{\sqrt{5}} = \frac{3}{\sqrt{5}}.$$