Lecture 4: The Cross Product

4.1 The cross product

There is no general way to define multiplication for vectors in $\mathbb{R}^n$, with the product also being a vector of the same dimension, which is useful for our current purposes. However, in the special case of $\mathbb{R}^3$ there is a product which we will find useful. One motivation for this product is to consider the following problem: Given two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, not parallel to one another, find a third vector $\mathbf{w} = (w_1, w_2, w_3)$ which is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$. Thus we want $\mathbf{w} \cdot \mathbf{x} = 0$ and $\mathbf{w} \cdot \mathbf{y} = 0$, which means we need to solve the equations

$$x_1 w_1 + x_2 w_2 + x_3 w_3 = 0$$
$$y_1 w_1 + y_2 w_2 + y_3 w_3 = 0$$

for w_1, w_2, and w_3. Multiplying the first equation by y_3 and the second by x_3 gives us

$$x_1 y_3 w_1 + x_2 y_3 w_2 + x_3 y_3 w_3 = 0$$
$$x_3 y_1 w_1 + x_3 y_2 w_2 + x_3 y_3 w_3 = 0.$$

Subtracting the second equation from the first, we have

$$(x_1 y_3 - x_3 y_1) w_1 + (x_2 y_3 - x_3 y_2) w_2 = 0.$$

One solution is given by setting

$$w_1 = x_2 y_3 - x_3 y_2$$
$$w_2 = -(x_1 y_3 - x_3 y_1) = x_3 y_1 - x_1 y_3.$$

Finally, from our first equation, we now have

$$x_3 w_3 = -x_1(x_2 y_3 - x_3 y_2) - x_2(x_3 y_1 - x_1 y_3) = x_1 x_3 y_2 - x_2 x_3 y_1,$$

from which we obtain the solution

$$w_3 = x_1 y_2 - x_2 y_1.$$

The choices made in arriving at these solutions are not unique, but they are the standard choices which define $\mathbf{w}$ as the *cross* or *vector product* of $\mathbf{x}$ and $\mathbf{y}$.

Definition   Given vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, the vector

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, \, x_3 y_1 - x_1 y_3, \, x_1 y_2 - x_2 y_1)$$

is called the *cross product*, or *vector product*, of $\mathbf{x}$ and $\mathbf{y}$.

Example   If $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, -1, 1)$, then
$$\mathbf{x} \times \mathbf{y} = (2 + 3, 3 - 1, -1 - 2) = (5, 2, -3).$$
Note that
$$\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = 5 + 4 - 9 = 0$$
and
$$\mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = 5 - 2 - 3 = 0,$$
showing that $\mathbf{x} \perp (\mathbf{x} \times \mathbf{y})$ and $\mathbf{y} \perp (\mathbf{x} \times \mathbf{y})$ as claimed. It is also interesting to note that
$$\mathbf{y} \times \mathbf{x} = (-3 - 2, 1 - 3, 2 + 1) = (-5, -2, 3) = -(\mathbf{x} \times \mathbf{y}).$$
This last calculation holds in general for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$.

Proposition   Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in $\mathbb{R}^3$ and α is any real number. Then
$$\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x}),$$
$$\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z}),$$
$$(\mathbf{x} + \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \times \mathbf{z}) + (\mathbf{y} \times \mathbf{z}),$$
$$\alpha(\mathbf{x} \times \mathbf{y}) = (\alpha\mathbf{x}) \times \mathbf{y} = \mathbf{x} \times (\alpha\mathbf{y}),$$
and
$$\mathbf{x} \times \mathbf{0} = \mathbf{0}.$$

Verification of these properties is straightforward. Also, notice that
$$\mathbf{i} \times \mathbf{j} = \mathbf{k},$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i},$$
and
$$\mathbf{k} \times \mathbf{i} = \mathbf{j};$$
that is, the cross product of two standard basis vectors is either the other standard basis vector or its negative. Moreover, note that in these cases the cross product points in the direction your thumb would point if you were to wrap the fingers of your right hand from the first vector to the second. This is in fact always true and results in what is known as the *right-hand rule* for the orientation of the cross product, as shown below. Hence given two vectors $\mathbf{x}$ and $\mathbf{y}$, we can always determine the direction of $\mathbf{x} \times \mathbf{y}$.
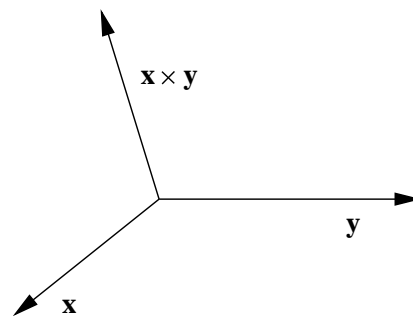
The right-hand rule

To complete our geometric identification of $\mathbf{x} \times \mathbf{y}$, we need to know only its length. Now if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$
\begin{aligned}
|\mathbf{x} \times \mathbf{y}|^2 &= (x_2 y_3 - x_3 y_2)^2 + (x_3 y_1 - x_1 y_3)^2 + (x_1 y_2 - x_2 y_1)^2 \\
&= x_2^2 y_3^2 - 2 x_2 x_3 y_2 y_3 + x_3^2 y_2^2 + x_3^2 y_1^2 - 2 x_1 x_3 y_1 y_3 + x_1^2 y_3^2 + x_1^2 y_2^2 \\
&\quad - 2 x_1 x_2 y_1 y_2 + x_2^2 y_1^2 \\
&= (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2) \\
&\quad - (2 x_2 x_3 y_2 y_3 + 2 x_1 x_3 y_1 y_3 + 2 x_1 x_2 y_1 y_2) \\
&= (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1 y_1 + x_2 y_2 + x_3 y_3)^2 \\
&= |\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \\
&= |\mathbf{x}|^2 |\mathbf{y}|^2 - (|\mathbf{x}||\mathbf{y}| \cos(\theta))^2 \\
&= |\mathbf{x}|^2 |\mathbf{y}|^2 (1 - \cos^2(\theta)) \\
&= |\mathbf{x}|^2 |\mathbf{y}|^2 \sin^2(\theta).
\end{aligned}
$$

Taking square roots, and noting that $\sin(\theta) \geq 0$ since, by the definition of the angle between two vectors, $0 \leq \theta \leq \pi$, we have the following result.

Proposition If θ is the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$, then

$$
|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}||\mathbf{y}| \sin(\theta).
$$

The last theorem has several interesting consequences. One of these comes from recognizing that if we draw a parallelogram with $\mathbf{x}$ and $\mathbf{y}$ as adjacent sides, as in the figure below, then the height of the parallelogram is $|\mathbf{y}| \sin(\theta)$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Hence the area of the parallelogram is $|\mathbf{x}||\mathbf{y}| \sin(\theta)$, which, by our proposition, is $|\mathbf{x} \times \mathbf{y}|$.
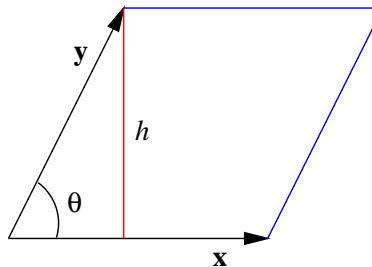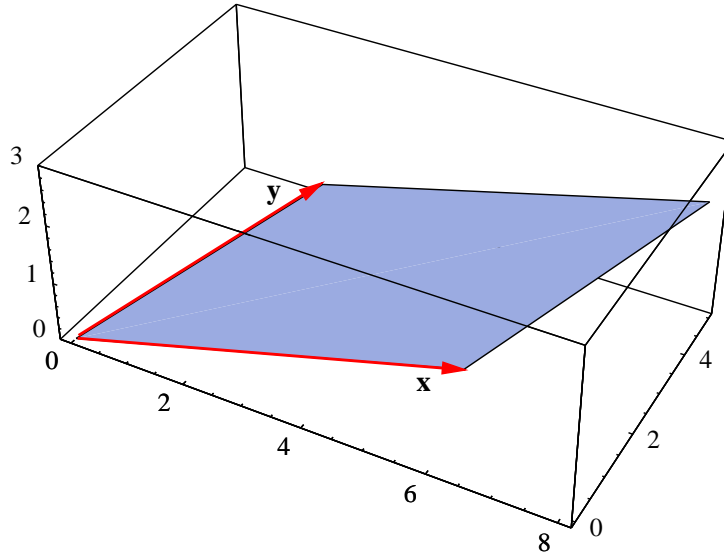

Height of the parallelogram is $h = |\mathbf{y}| \sin(\theta)$

Proposition Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^3$. Then the area of the parallelogram which has $\mathbf{x}$ and $\mathbf{y}$ for adjacent sides is $|\mathbf{x} \times \mathbf{y}|$.

Example Consider the parallelogram P with vertices at $(0, 0, 0)$, $(6, 1, 1)$, $(8, 5, 2)$, and $(2, 4, 1)$. Two adjacent sides are specified by the vectors $\mathbf{x} = (6, 1, 1)$ and $\mathbf{y} = (2, 4, 1)$, so the area of P is

$$
|\mathbf{x} \times \mathbf{y}| = |(1 - 4, 2 - 6, 24 - 2)| = |(-3, -4, 22)| = \sqrt{509}.
$$

Parallelogram with vertices at $(0,0,0)$, $(6,1,1)$, $(8,5,2)$, and $(2,4,1)$

Example   Consider the parallelogram P in the plane with vertices at $(1,1,)$, $(3,2)$, $(4,4)$, and $(2,3)$. Two adjacent sides are given by the vectors from $(1,1)$ to $(3,2)$, that is
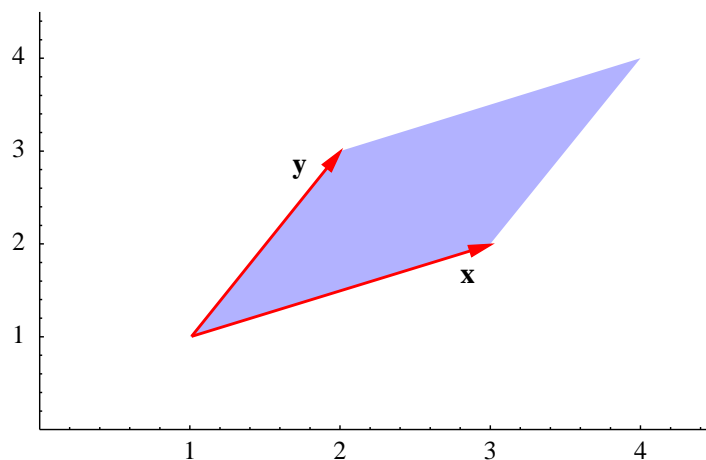
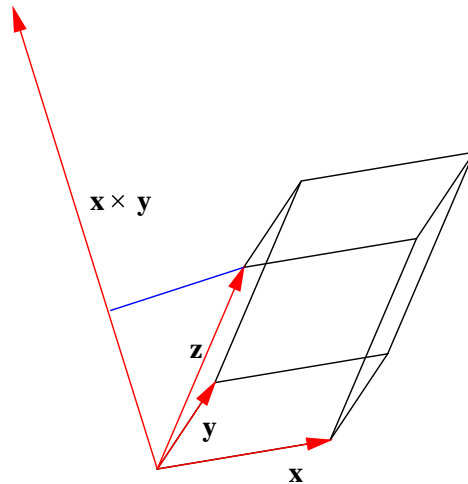$$\mathbf{x} = (3,2) - (1,1) = (2,1),$$

and from $(1,1)$ to $(2,3)$, that is,

$$\mathbf{y} = (2,3) - (1,1) = (1,2).$$

However, since these vectors are in $\mathbb{R}^2$, not in R^3, we cannot compute their cross product. To get around this, we consider the vectors $\mathbf{w} = (2,1,0)$ and $\mathbf{v} = (1,2,0)$ which are adjacent sides of the same parallelogram viewed as lying in $\mathbb{R}^3$. Then the area of P is given by

$$|\mathbf{w} \times \mathbf{v}| = |(0,0,4-1)| = |(0,0,3)| = 3.$$

Parallelogram with vertices at $(1,1)$, $(3,2)$, $(4,4)$, and $(2,3)$

Parallelepiped with adjacent edges $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$

It is easy to extend the result of the previous theorem to computing the volume of a parallelepiped in $\mathbb{R}^3$. Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are adjacent edges of parallelepiped P. Then the volume V of P is $|\mathbf{x} \times \mathbf{y}|$, which is the area of the base, multiplied by the height of P, which is the length of the projection of $\mathbf{z}$ onto $\mathbf{x} \times \mathbf{y}$. Since the latter is equal to

$$\left| \mathbf{z} \cdot \frac{\mathbf{x} \times \mathbf{y}}{|\mathbf{x} \times \mathbf{y}|} \right|,$$

we have

$$V = |\mathbf{x} \times \mathbf{y}| \left| \mathbf{z} \cdot \frac{\mathbf{x} \times \mathbf{y}}{|\mathbf{x} \times \mathbf{y}|} \right| = |\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})|.$$

Proposition   The volume of a parallelepiped with adjacent edges $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ is $|\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})|$.

Definition   Given three vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^3$, the quantity $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$ is called the *scalar triple product* of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}$.

Example   Let $\mathbf{x} = (1, 4, 1)$, $\mathbf{y} = (-3, 1, 1)$, and $\mathbf{z} = (0, 1, 5)$ be adjacent edges of parallelepiped P. Then

$$\mathbf{x} \times \mathbf{y} = (4 - 1, -3 - 1, 1 + 12) = (3, -4, 13),$$
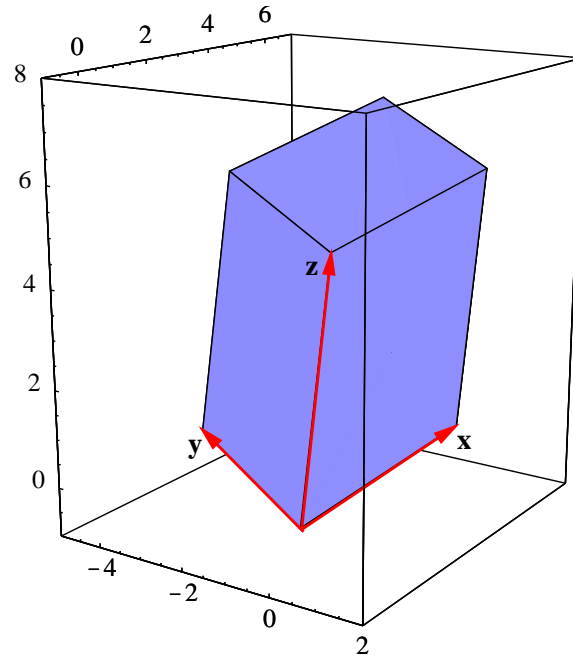
so

$$\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) = 0 - 4 + 65 = 61.$$

Hence the volume of P is 61.

The next result follows from the fact that the angle between parallel vectors is either 0 or π.

Proposition   Nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ are parallel if and only if $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.

Parallelepiped with adjacent edges $\mathbf{x} = (1, 4, 1)$, $\mathbf{y} = (-3, 1, 1)$, $\mathbf{z} = (0, 1, 5)$

Note that, in particular, for any vector $\mathbf{x}$ in $\mathbb{R}^3$, $\mathbf{x} \times \mathbf{x} = \mathbf{0}$.

You may find familiarity with determinants of 3×3 matrices helpful in computing cross products. Recall that the determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and the determinant of a 3×3 matrix is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & b_1 \\ c_1 & c_2 \end{vmatrix}.$$

Symbolically, we may remember the formula for the cross product of $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ as

$$\begin{aligned} \mathbf{x} \times \mathbf{y} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &= (x_2 y_3 - x_3 y_2)\mathbf{i} - (x_1 y_3 - x_3 y_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k} \\ &= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1). \end{aligned}$$