Lecture 27: Green's Theorem

27.1 Green's Theorem on a rectangle

Suppose $F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a continuous vector field defined on a closed rectangle $D = [a,b] \times [c,d]$. Let ∂D be the boundary of D oriented in the counterclockwise direction. Let C_1, C_2, C_3, and C_4 be the sides of ∂D in counterclockwise order, starting with the bottom. We parametrize C_1 by $\alpha(t) = (t,c)$, $a \leq t \leq b$; C_2 by $\beta(t) = (b,t)$, $c \leq t \leq d$; $-C_3$ by $\gamma(t) = (t,d)$, $a \leq t \leq b$; and $-C_4$ by $\delta(t) = (a,t)$, $c \leq t \leq d$. Then

$$\begin{aligned}
\int_{\partial D} F \cdot d\mathbf{r} &= \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r} + \int_{C_3} F \cdot d\mathbf{r} + \int_{C_4} F \cdot d\mathbf{r} \\
&= \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r} - \int_{-C_3} F \cdot d\mathbf{r} - \int_{-C_4} F \cdot d\mathbf{r} \\
&= \int_a^b P(t,c)dt + \int_c^d Q(b,t)dt - \int_a^b P(t,d)dt - \int_c^d Q(a,t)dt \\
&= -\int_a^b (P(t,d) - P(t,c))dt + \int_c^d (Q(b,t) - Q(a,t))dt \\
&= -\int_a^b \int_c^d \frac{\partial}{\partial y} P(t,y)dydt + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,t)dxdt \\
&= -\int_a^b \int_c^d \frac{\partial}{\partial y} P(x,y)dydx + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x,y)dxdy \\
&= \int_a^b \int_c^d \left(\frac{\partial}{\partial x} Q(x,y) - \frac{\partial}{\partial y} P(x,y) \right) dydx \\
&= \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.
\end{aligned}$$

This result is a special case of *Green's Theorem.*

Example We will evaluate

$$\int_C x^2 y dx + xy dy,$$

where C is the rectangle with vertices $(0,0)$, $(3,0)$, $(3,1)$, and $(0,1)$, oriented in the counterclockwise direction. Using Green's Theorem,

$$\begin{aligned}
\int_C x^2 y dx + xy dy &= \int_0^3 \int_0^1 (y - x^2) dy dx \\
&= \int_0^3 \left(\frac{1}{2} - x^2 \right) dx \\
&= \frac{3}{2} - 9 \\
&= -\frac{15}{2}.
\end{aligned}$$

27.2 Green's Theorem

Definition A *simple closed* curve in $\mathbb{R}^n$ is a curve which is closed and does not intersect itself. The *positive orientation* of a simple closed curve is the counterclockwise orientation.

Green's Theorem Suppose $F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a continuous vector field defined on a region D in $\mathbb{R}^2$. Moreover, suppose P and Q have continuous partial derivatives and that the boundary ∂D is a simple closed curve with positive orientation. Then

$$\int_{\partial D} P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Example Let C be the triangle with vertices at $(0,0)$, $(1,1)$, and $(0,1)$, with positive orientation. Then

$$\begin{aligned} \int_C xy dx - x dy &= \int_0^1 \int_0^y (-1 - x) dx dy \\ &= -\int_0^1 \int_0^y (1 + x) dx dy \\ &= -\int_0^1 \left(y + \frac{y^2}{2} \right) dy \\ &= -\left(\frac{1}{2} + \frac{1}{6} \right) \\ &= -\frac{2}{3}. \end{aligned}$$

Note that each of the vector fields

$$F(x,y) = (0, x),$$

$$F(x,y) = (-y, 0),$$

and

$$F(x,y) = \frac{1}{2}(-y, x)$$

has the property that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

It follows that if A is the area of a region D which satisfies the conditions of Green's theorem, then

$$A = \int\int_D dA = \int_{\partial D} x dy,$$

$$A = \int\int_D dA = -\int_{\partial D} y dx,$$

and

$$A = \int\int_D dA = \frac{1}{2}\int_{\partial D} xdy - ydx.$$

Example Let E be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We may parametrize E, in the counterclockwise direction, using $\varphi(t) = (a\cos(t), b\sin(t))$, $0 \le t \le 2\pi$. If A is the area of the region D enclosed by E, then

$$\begin{aligned} A &= \int\int_D dA \\ &= \frac{1}{2}\int_E xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi} (-b\sin(t), a\cos(t)) \cdot (-a\sin(t), b\cos(t)dt \\ &= \frac{1}{2}\int_0^{2\pi} (ab\sin^2(t) + ab\cos^2(t))dt \\ &= \frac{1}{2}\int_0^{2\pi} abdt \\ &= \pi ab. \end{aligned}$$

Note that if $F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, P and Q have continuous partial derivatives on an open simply connected region D, and

$$\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial x}Q(x,y)$$

for all (x,y) in D, it now follows, by Green's Theorem, that

$$\int_C F \cdot d\mathbf{r} = 0$$

for any simply closed curve in C. It follows that F is a conservative vector field.

Notation: If C is a closed curve with orientation in the counterclockwise direction, the line integral

$$\int_C F \cdot d\mathbf{r}$$

may be denoted

$$\oint_C F \cdot d\mathbf{r}.$$