Lecture 26: Conservative Vector Fields

26.1 The line integral of a conservative vector field

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and the vector field $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Let $F(\mathbf{x}) = \nabla f(\mathbf{x})$. Then F is a conservative vector field. If $\varphi : [a, b] \to \mathbb{R}^n$ is a smooth parametrization of a curve C, then

$$\begin{aligned}\int_C F \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= \int_a^b \nabla f(\varphi(t)) \cdot \varphi'(t) dt \\ &= \int_a^b \frac{d}{dt} f(\varphi(t)) dt \\ &= f(\varphi(t))\Big|_a^b \\ &= f(\varphi(b)) - f(\varphi(a)) \\ &= f(\mathbf{b}) - f(\mathbf{a}),\end{aligned}$$

where $\mathbf{a} = \varphi(a)$ is the *initial point* of C and $\mathbf{b} = \varphi(b)$ is the *terminal point* of C.

Example The vector field

$$F(x, y, z) = -\frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}(x, y, z)$$

is a conservative vector field with potential

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

Hence if C is a curve with initial point $(1, 0, 0)$ and terminal point $(-2, 2, 3)$, then

$$\int_C F \cdot d\mathbf{r} = f(-2, 2, 1) - f(1, 0, 0) = \frac{1}{3} - 1 = -\frac{2}{3}.$$

26.2 Path independence

Definition Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field. If for any two curves C_1 and C_2 in the domain of F with the same initial and terminal points we have

$$\int_{C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r},$$

then we say $\int_C F \cdot d\mathbf{r}$ is *independent of path.*

Definition A curve C whose initial and terminal points are the same is called a *closed curve.*

Proposition $\int_C F \cdot d\mathbf{r}$ is independent of path if and only if $\int_C F \cdot d\mathbf{r} = 0$ for every closed path C in the domain of F.

Proof First suppose $\int_C F \cdot d\mathbf{r}$ is independent of path and let C be a closed curve. Let $\mathbf{a}$ and $\mathbf{b}$ be two points on C. Let C_1 be the part of C from $\mathbf{a}$ to $\mathbf{b}$ and let C_2 be the part of C from $\mathbf{b}$ to $\mathbf{a}$. Then

$$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} - \int_{-C_2} F \cdot d\mathbf{r} = 0,$$

where the final equality follows from the fact that C_1 and $-C_2$ have the same initial and terminal points. Now suppose $\int_C F \cdot d\mathbf{r} = 0$ for any closed curve C. Let C_1 and C_2 be two curves, both with initial point $\mathbf{a}$ and terminal point $\mathbf{b}$. Let C be the closed curve obtained by first traversing C_1 and then traversing $-C_2$. Then

$$0 = \int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{-C_2} F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} - \int_{C_2} F \cdot d\mathbf{r},$$

and so

$$\int_{C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r}.$$

Definition We say a region D in $\mathbb{R}^n$ is *path connected*, or *connected*, if for any two points $\mathbf{a}$ and $\mathbf{b}$ in D there exists a path from $\mathbf{a}$ to $\mathbf{b}$ which lies entirely within D.

Proposition Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field defined on an open connected region D. If $\int_C F \cdot d\mathbf{r}$ is independent of path in D, then F is a conservative vector field.

Proof We will prove the proposition for $n = 2$. Let (a, b) be a point in D. We define a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x, y) = \int_C F \cdot d\mathbf{r},$$

where C is a path with initial point (a, b) and terminal point (x, y). Note that, because of path independence, the value of $f(x, y)$ depends only on (x, y), and not on the choice for C. Let

$$F(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}.$$

Now

$$\frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

Let C be a path from (a, b) to (x, y) and let C_1 be a horizontal path from (x, y) to $(x+h, y)$. Then

$$\begin{aligned} f(x+h, y) - f(x, y) &= \int_C F \cdot d\mathbf{r} + \int_{C_1} F \cdot d\mathbf{r} - \int_C F \cdot d\mathbf{r} \\ &= \int_{C_1} F \cdot d\mathbf{r} \end{aligned}.$$

If we parametrize C_1 by $\varphi(t) = (x + t, y)$, $0 \leq t \leq h$, then $\varphi'(t) = (1, 0)$ and

$$\int_C F \cdot d\mathbf{r} = \int_0^h (P(x+t, y), Q(x+t, y)) \cdot (1, 0) dt = \int_0^h P(x+t, y) dt.$$

Hence

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= \lim_{h \to 0} \frac{\int_0^h P(x+t, y) dt}{h} \\ &= \lim_{h \to 0} P(x+h, y) \text{ (using l'Hôpital's rule)} \\ &= P(x, y). \end{aligned}$$

A similar calculation shows that

$$\frac{\partial}{\partial y} f(x, y) = Q(x, y),$$

and so $\nabla f(x, y) = F(x, y)$.

Proposition If $F(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field and P and Q have continuous partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Proof Let f be a potential function for F. Then

$$\frac{\partial P}{\partial y} = \frac{\partial^2}{\partial y \partial x} f(x, y) = \frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial Q}{\partial x}.$$

Example The vector field $F(x, y) = (x^2 y, xy^3)$ is not conservative since

$$\frac{\partial}{\partial y}(x^2 y) = x^2$$

and

$$\frac{\partial}{\partial x}(xy^3) = y^3.$$

Example Let

$$F(x,y) = \frac{1}{x^2+y^2}(-y,x).$$

If we let

$$P(x,y) = -\frac{y}{x^2+y^2}$$

and

$$Q(x,y) = \frac{x}{x^2+y^2},$$

then

$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Hence

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

However, F is not a conservative vector field. For example, if C is the unit circle centered at the origin parametrized by $\varphi(t) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$, then

$$\int_C F \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt = \int_0^{2\pi} dt = 2\pi.$$

Hence $\int_C F \cdot d\mathbf{r} \neq 0$, and so F cannot be conservative.

It follows from the previous example that the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

although a necessary condition for a vector field $F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ to be conservative, is not a sufficient condition. The problem in the previous example turns out to be the nature of the domain of the vector field. Note that the closed curve C which yields the nonzero line integral contains the origin, a point at which F is not defined. We will see in the next section that if P and Q have continuous partial derivatives on an open region D,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

for all (x,y) in D, and D has the property that for any closed curve C in D, all points "inside" C lie in D (that is, D has no "holes"), then F is conservative. We say that such a region D is *simply connected*.

Example Suppose

$$F(x, y) = (2xy + 2x, x^2 - 6y).$$

If we let

$$P(x, y) = 2xy + 2x$$

and

$$Q(x, y) = x^2 - 6y,$$

then

$$\frac{\partial}{\partial y} P(x, y) = 2x = \frac{\partial}{\partial x} Q(x, y).$$

Since the domain of F is all of $\mathbb{R}^2$, it follows that F is conservative. That is, there exists a scalar field f for which

$$\frac{\partial}{\partial x} f(x, y) = 2xy + 2x$$

and

$$\frac{\partial}{\partial y} f(x, y) = x^2 - 6y.$$

From the first of these two equations, we have

$$f(x, y) = \int (2xy + 2x) dx = x^2 y + x^2 + g(y)$$

for some function g which depends only on y. Now we must have

$$x^2 - 6y = \frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} (x^2 y + x^2 + g(y)) = x^2 + g'(y),$$

from which it follows that

$$g'(y) = -6y.$$

Hence

$$g(y) = -3y^2 + c$$

for some constant c. Thus for any constant c the function

$$f(x, y) = x^2 y + x^2 - 3y^2 + c$$

is a potential function for the vector field F.