Lecture 25: Line Integrals

25.1 The line integral of a scalar field

Suppose $\varphi : [a, b] \to \mathbb{R}^n$ is a smooth parametrization of a curve C and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous scalar field. Let

$$s = \int_a^t |\varphi'(u)| du.$$

Then s is the length of the piece of C extending from $\varphi(a)$ to $\varphi(t)$. Note that

$$\frac{ds}{dt} = |\varphi'(t)|.$$

We now define

$$\int_C f(\mathbf{x}) ds = \int_a^b f(\varphi(t)) \frac{ds}{dt} dt = \int_a^b f(\varphi(t)) |\varphi'(t)| dt,$$

which we call the *line*, or *path*, *integral of* f *along* C. In particular, if $n = 3$ and $\varphi(t) = (x(t), y(t), z(t))$, then

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt;$$

if $n = 2$ and $\varphi(t) = (x(t), y(t))$, then

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Geometrically, we may think of the latter integral as the area of a "fence" with base along the curve C and height given by $f(x, y)$.

Example Let C be the circular helix parametrized by

$$\varphi(t) = (\cos(t), \sin(t), t)$$

for $0 \leq t \leq 2\pi$ and let

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Then

$$|\varphi(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

and

$$f(\varphi(t)) = \cos^2(t) + \sin^2(t) + t^2 = 1 + t^2.$$

Hence

$$\begin{aligned}\int_C f(x,y,z)ds &= \int_0^{2\pi}(1+t^2)\sqrt{2}dt\\ &= \sqrt{2}\left(2\pi + \frac{t^3}{3}\Big|_0^{2\pi}\right)\\ &= \sqrt{2}\left(2\pi + \frac{8\pi^3}{3}\right)\\ &= 2\pi\sqrt{2}\left(1+\frac{4\pi^2}{3}\right).\end{aligned}$$

Example We will evaluate $\int_C xyds$ where C is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Let C_1 be the line from $(0,0)$ to $(1,0)$, C_2 the line from $(1,0)$ to $(1,1)$, and C_3 the line from $(1,1)$ to $(0,0)$. Now $\alpha(t)=(t,0)$, $0\le t\le 1$, parametrizes C_1, $\beta(t)=(1,t)$, $0\le t\le 1$, parametrizes C_2, and $\gamma(t)=(1-t,1-t)$, $0\le t\le 1$, parametrizes C_3, so

$$\int_{C_1} xyds = \int_0^1 0dt = 0,$$

$$\int_{C_2} xyds = \int_0^1 tdt = \frac{1}{2},$$

and

$$\int_{C_3} xyds = \int_0^1 (1-t)^2\sqrt{2}dt = -\frac{\sqrt{2}(1-t)^3}{3}\Big|_0^1 = \frac{\sqrt{2}}{3}.$$

Thus

$$\int_C xyds = \int_{C_1} xyds + \int_{C_2} xyds + \int_{C_3} xyds = \frac{1}{2}+\frac{\sqrt{2}}{3} = \frac{3+2\sqrt{2}}{6}.$$

Note that we could have parametrized C_3 by $\delta(t)=(t,t)$, $0\le t\le 1$, which would give the same result:

$$\int_{C_3} t^2\sqrt{2}dt = \frac{\sqrt{2}}{3}.$$

25.2 The line integral of a vector field

In physics, if a force of constant magnitude F acts to move an object a distance d along a line, then $W=Fd$ is called the *work* done by the force. Slightly more generally, if an object moves along a vector $\mathbf{d}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ in the presence of a constant vector force $\mathbf{F}$, then the work W done by F is the product of the component of $\mathbf{F}$ in the direction of $\mathbf{d}$ and the length of $\mathbf{d}$. That is,

$$W = \left(\mathbf{F}\cdot\frac{\mathbf{d}}{|\mathbf{d}|}\right)|\mathbf{d}| = \mathbf{F}\cdot\mathbf{d}.$$

Now suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field, $\varphi : [a, b] \to \mathbb{R}^n$ is a smooth parametrization of a curve C, and W is the work done by the force F in moving an object along the curve C from $\varphi(a)$ to $\varphi(b)$. If we divide $[a, b]$ into n subintervals of equal length Δt, then

$$W \approx \sum_{i=0}^{n-1} F(\varphi(t_i)) \cdot (\varphi(t_{i+1}) - \varphi(t_i)),$$

an approximation which should improve as n increases. In fact, we should have

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=0}^{n-1} F(\varphi(t_i)) \cdot (\varphi(t_{i+1}) - \varphi(t_i)) \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} F(\varphi(t_i)) \cdot \frac{\varphi(t_{i+1}) - \varphi(t_i)}{\Delta t} \Delta t \\ &= \int_a^b F(\varphi(t)) \cdot \varphi'(t) dt. \end{aligned}$$

We call this integral the *line integral of F along C*, and denote it

$$\int_C F \cdot d\mathbf{r}$$

(the $\mathbf{r}$ comes from thinking of the curve C as having vector equation $\mathbf{r} = \varphi(t)$). That is,

$$\int_C F \cdot d\mathbf{r} = \int_a^b F(\varphi(t)) \cdot \varphi'(t) dt.$$

Example We will evaluate $\int_C F \cdot d\mathbf{r}$ where C is the unit circle parametrized by $\varphi(t) = (\cos(t), \sin(t))$, $0 \le t \le 2\pi$, and F is the vector field $F(x, y) = (-y, x)$. Then

$$\int_C F \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt = \int_0^{2\pi} dt = 2\pi.$$

If we let $-C$ be C parametrized in the reverse direction by $\varphi(t) = (\sin(t), \cos(t))$, then

$$\int_{-C} F \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos(t), \sin(t)) \cdot (\cos(t), -\sin(t)) dt = -\int_0^{2\pi} dt = -2\pi.$$

In general, if $-C$ is the curve C parametrized in the reverse direction, then

$$\int_{-C} F \cdot d\mathbf{r} = -\int_C F \cdot d\mathbf{r}.$$

Notation: If

$$F(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n)), \ldots, f_n(x_1, x_2, \ldots, x_n)$$

and

$$\varphi(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

parametrizes a curve C for $a \leq t \leq b$, then

$$\begin{aligned}
\int_C F \cdot d\mathbf{r} &= \int_a^b (f_1(x_1(t), x_2(t), \ldots, x_n(t)), \ldots, f_n(x_1(t), x_2(t), \ldots, x_n(t))) \\
&\qquad \cdot \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}\right) dt \\
&= \int_a^b \left(f_1(x_1(t), x_2(t), \ldots, x_n(t)) \frac{dx_1}{dt} + f_2(x_1(t), x_2(t), \ldots, x_n(t)) \frac{dx_2}{dt} + \cdots \right. \\
&\qquad \left. + f_n(x_1(t), x_2(t), \ldots, x_n(t)) \frac{dx_n}{dt} \right) dt \\
&= \int_C f_1(x_1, x_2, \ldots, x_n) dx_1 + f_2(x_1, x_2, \ldots, x_n) dx_2 + \cdots \\
&\qquad + f_n(x_1, x_2, \ldots, x_n) dx_n.
\end{aligned}$$

In particular, for $n = 3$, if

$$F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k},$$

we may write

$$\int_C F \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz,$$

and, for $n = 2$, if

$$F(x, y) = (P(x, y), Q(x, y)) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j},$$

we may write

$$\int_C F \cdot d\mathbf{r} = \int_C P dx + Q dy.$$

Example We will evaluate

$$\int_C x^2 y dx + 3x dy,$$

where C is the segment of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$. We may parametrize C by $\varphi(t) = (t, t^2)$, $0 \le t \le 2$, so

$$\begin{aligned}\int_C x^2 y dx + 3x dy &= \int_0^2 (t^4, 3t) \cdot (1, 2t) dt \\ &= \int_0^2 (t^4 + 6t^2) dt \\ &= \frac{32}{5} + 16 \\ &= \frac{112}{5}.\end{aligned}$$

Example We will evaluate

$$\int_C x^2 dz,$$

where C is the curve in $\mathbb{R}^3$ parametrized by $\varphi(t) = (t, \cos(t), \sin(t))$, $0 \le t \le 2\pi$. Then $\varphi'(t) = (1, -\sin(t), \cos(t))$, so

$$\begin{aligned}\int_C x^2 dz &= \int_0^{2\pi} (0, 0, t^2) \cdot (1, -\sin(t), \cos(t)) dt \\ &= \int_0^{2\pi} t^2 \cos(t) dt \\ &= t^2 \sin(t)\Big|_0^{2\pi} - \int_0^{2\pi} 2t \sin(t) dt \\ &= 2t \cos(t)\Big|_0^{2\pi} - \int_0^{2\pi} 2\cos(t) dt \\ &= 4\pi - 2\sin(t)\Big|_0^{2\pi} \\ &= 4\pi.\end{aligned}$$

More on notation: Suppose $\varphi : [a, b] \to \mathbb{R}^n$ parametrizes a curve C and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field. Let

$$T(t) = \frac{\varphi'(t)}{|\varphi'(t)|}$$

be the unit tangent vector to C at $\varphi(t)$. Then

$$\int_C F \cdot d\mathbf{r} = \int_a^b F(\varphi(t)) \cdot \varphi'(t) dt = \int_a^b F(\varphi(t)) \cdot T(t) |\varphi'(t)| dt = \int_C F \cdot T ds.$$

The latter integral is a common notation for the line integral of a vector field F along a curve C.