Lecture 24: Vector Fields

24.1 Vector fields

Definition A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a *vector field.*

If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, then, for any $\mathbf{x}$ in $\mathbb{R}^n$,

$$F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for some functions $f_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, 2, \ldots, n$, the *component*, or *coordinate, functions* of f. In contrast to the vector field F, we call the functions f_1, f_2, $\ldots$, f_n *scalar fields.*

Definition We say a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is *continuous* at a point $\mathbf{a}$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} F(\mathbf{x}) = F(\mathbf{a}).$$

Proposition Suppose a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ has coordinate functions f_1, f_2, $\ldots$, f_n. Then F is continuous at a if and only if f_k is continuous at $\mathbf{a}$ for $k = 1, 2, \ldots, n$.

Example The vector field

$$\begin{aligned} F(x,y,z) &= -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x,y,z) \\ &= -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{i} - \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{j} - \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{k} \end{aligned}$$

is continuous at every point in $\mathbb{R}^3$ except $(0,0,0)$. Note that for any $(x,y,z) \neq (0,0,0)$,

$$|F(x,y,z)| = \frac{|(x,y,z)|}{|(x,y,z)|^3} = \frac{1}{|(x,y,z)|^2}.$$

Hence we may picture $F(x,y,z)$ as a vector pointing from (x,y,z) towards the origin with length equal to the reciprocal of the square of the distance from (x,y,z) to the origin. Multiplied by the appropriate constants, this vector field could represent the gravitational force field of a point mass at the origin.

Definition Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field. If there exists a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ such that $F(\mathbf{x}) = \nabla f(\mathbf{x})$ for all $\mathbf{x}$ in the domain of F, then we call f a *potential function* for F and we say that F is a *conservative vector field.*

Example Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

Then

$$\nabla f(x,y,z) = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x,y,z).$$

Hence the vector field

$$F(x,y,z) = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x,y,z)$$

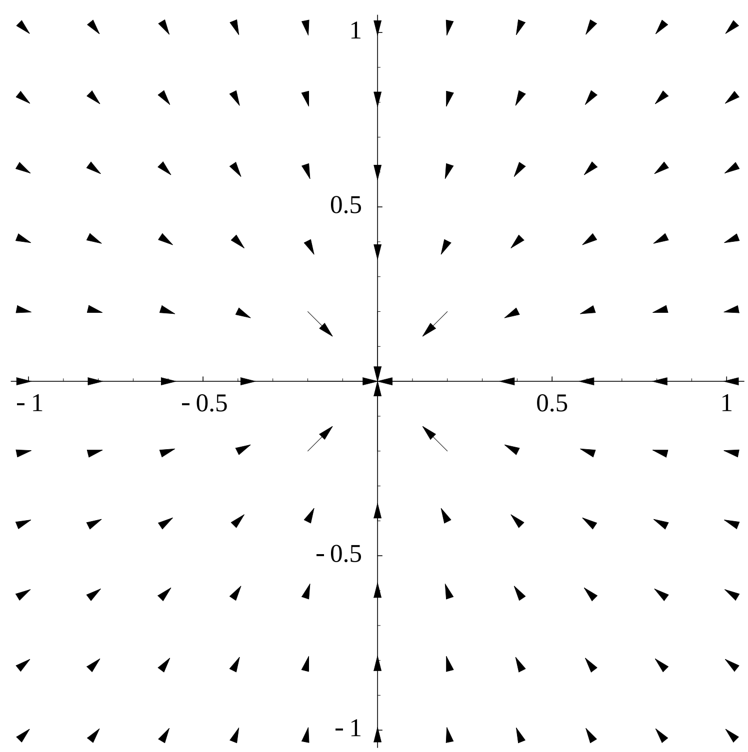
of the previous example is a conservative vector field with potential f.

24.2 Picturing a vector field

Example The following plot shows the result of plotting a scaled version of vectors from the vector field

$$F(x,y) = -\frac{1}{x^2+y^2}(x,y) = -\frac{x}{x^2+y^2}\mathbf{i} - \frac{y}{x^2+y^2}\mathbf{j}$$

on a grid in the xy-plane.



The vector field $F(x,y) = -\frac{1}{x^2+y^2}(x,y)$

Note that this is the gradient vector field of the potential

$$f(x,y) = -\frac{1}{2}\ln(x^2+y^2).$$

Example The following plot shows the result of plotting a scaled version of vectors from the vector field

$$F(x,y) = \frac{1}{\sqrt{x^2+y^2}}(-y,x)$$

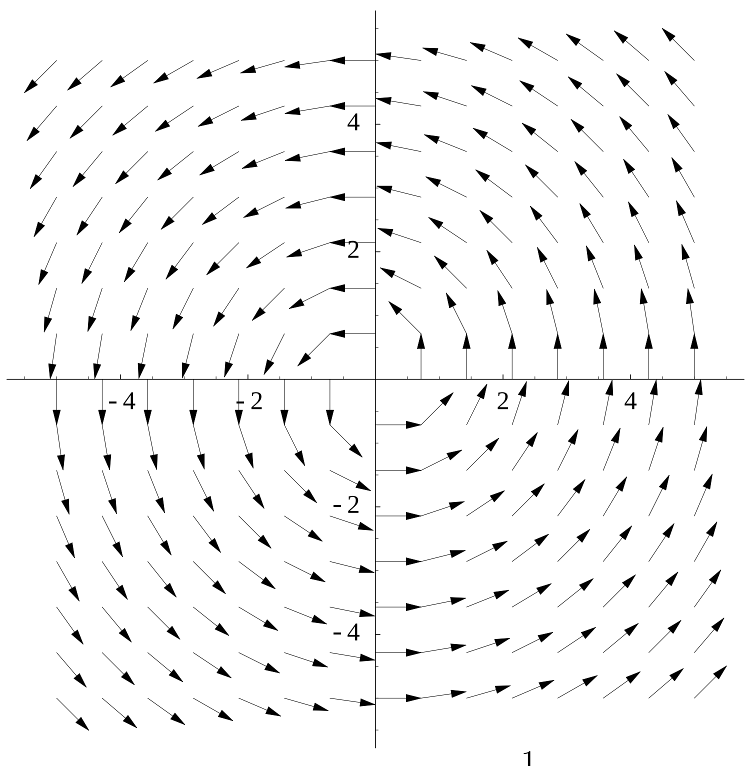
on a grid in the xy-plane. Note that

$$|F(x,y)| = \frac{1}{\sqrt{x^2+y^2}}|(-y,x)| = \frac{\sqrt{y^2+x^2}}{\sqrt{x^2+y^2}} = 1,$$

which explains why the vectors are all the same length, and

$$(x,y)\cdot F(x,y) = \frac{-yx+xy}{\sqrt{x^2+y^2}} = 0,$$

showing that $F(x,y)$ is orthogonal to (x,y), or, in other words, $F(x,y)$ is tangent to the circle centered at the origin passing through (x,y).



The vector field $F(x,y) = -\frac{1}{\sqrt{x^2+y^2}}(-y,x)$