Lecture 23: Cylindrical and Spherical Coordinates

23.1 Cylindrical coordinates

If P is a point in 3-space with Cartesian coordinates (x, y, z) and (r, θ) are the polar coordinates of (x, y), then (r, θ, z) are the *cylindrical coordinates* of P. If $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous on a region in space described by D in Cartesian coordinates and by T in cylindrical coordinates, then

$$\int\int\int_D f(x, y, z)dxdydz = \int\int\int_T f(r\cos(\theta), r\sin(\theta), z)rdrd\theta dz.$$

Example If V is the volume of the region D bounded by the two surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$, then, changing to cylindrical coordinates,

$$V = \int\int\int_D dxdydz = \int_0^1 \int_0^{2\pi} \int_{r^2}^{2-r^2} rdzd\theta dr = \int_0^1 \int_0^{2\pi} 2(r - r^3)d\theta dr = \pi,$$

where the evaluation of the final double integral follows as in a previous example.

23.2 Spherical coordinates

We may describe a point P in 3-space using coordinates (ρ, θ, ϕ) where ρ is the distance from P to the origin, θ is the polar coordinate angle for the projection of P onto the horizontal plane, and ϕ is the angle between the line from the origin to P and the vertical axis. Note that we may describe all of 3-space with $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \rho \leq \pi$. We call (ρ, θ, ϕ) the *spherical coordinates* of P.

If P has Cartesian coordinates (x, y, z), then

$$\begin{aligned}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
\tan(\theta) &= \frac{y}{x} \\
\cos(\phi) &= \frac{z}{\sqrt{x^2 + y^2 + z^2}}.
\end{aligned}$$

Conversely, if P has spherical coordinates (ρ, θ, ϕ), then

$$\begin{aligned}
x &= \rho\sin(\phi)\cos(\theta) \\
y &= \rho\sin(\phi)\sin(\theta) \\
z &= \rho\cos(\phi).
\end{aligned}$$

Example A point P with Cartesian coordinates $(2, -2, 1)$ has spherical coordinates

$$\begin{aligned}
\rho &= 3 \\
\theta &= \frac{7\pi}{4} \\
\phi &= \cos^{-1}\left(\frac{1}{3}\right) = 1.2310,
\end{aligned}$$

where the final result is rounded to 4 decimal places.

Example A point P with spherical coordinates $\left(4, \frac{\pi}{3}, \frac{3\pi}{4}\right)$ has Cartesian coordinates

$$\begin{aligned}
x &= \sqrt{2} \\
y &= \sqrt{6} \\
z &= -2\sqrt{2}.
\end{aligned}$$

Now consider a spherical "rectangle" S of dimensions $\Delta\rho$, $\Delta\theta$, and $\Delta\phi$. That is, for some fixed ρ, θ, and ϕ, let

$$S = \{(\alpha, \beta, \gamma) : \rho \le \alpha \le \rho + \Delta\rho, \theta \le \beta \le \theta + \Delta\theta, \phi \le \gamma \le \phi + \Delta\phi\}.$$

If V is the volume of S, then

$$V \approx \Delta\rho \times \rho\Delta\phi \times \rho\sin(\phi)\Delta\theta = \rho^2 \sin(\phi)\Delta\rho\Delta\theta\Delta\phi.$$

It follows that if $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous on a region in 3-space, described by D in Cartesian coordinates and by T in spherical coordinates, then

$$\int\int\int_D f(x, y, z)dxdydz = \\
\int\int\int_T f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2 \sin(\phi)d\rho d\theta d\rho.$$

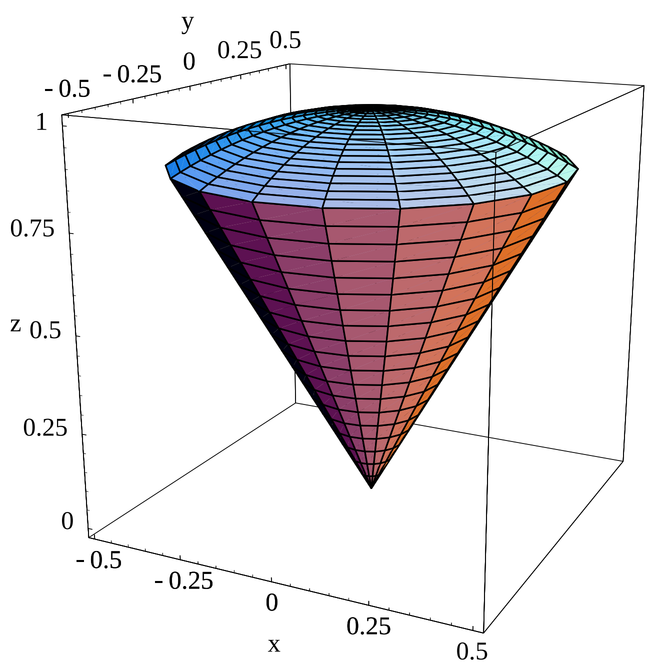
Example Let V be the volume of the region D bounded by the sphere of radius r centered at the origin and the cone with vertex at the origin whose sides make an angle α with the positive z-axis. Then

$$\begin{aligned}
V &= \int\int\int_D dV \\
&= \int_0^{\alpha}\int_0^{r}\int_0^{2\pi} \rho^2 \sin(\phi)d\theta d\rho d\phi
\end{aligned}$$

$$= 2\pi \int_0^\alpha \int_0^r \rho^2 \sin(\phi) d\rho d\phi$$
$$= 2\pi \int_0^\alpha \sin(\phi) \frac{\rho^3}{3}\Big|_0^r d\phi$$
$$= \frac{2\pi r^3}{3} \int_0^\alpha \sin(\phi) d\phi$$
$$= \frac{2\pi r^3}{3}(-\cos(\phi)\big|_0^\alpha$$
$$= \frac{2\pi r^3}{3}(1 - \cos(\alpha)).$$

Note that if $\alpha = \pi$, then D is the entire sphere of radius r, and we have the formula for the volume of a sphere of a radius r:

$$V = \frac{4}{3}\pi r^3.$$



Region bound by a sphere of radius 1 and a cone with an angle of $\frac{\pi}{6}$ with the z-axis